Context-Free Trees

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Department of Mathematics
University of Manchester

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29 July 2026

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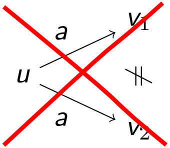

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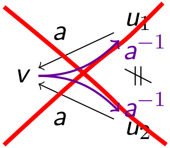

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End-Cones and End-Isomorphisms

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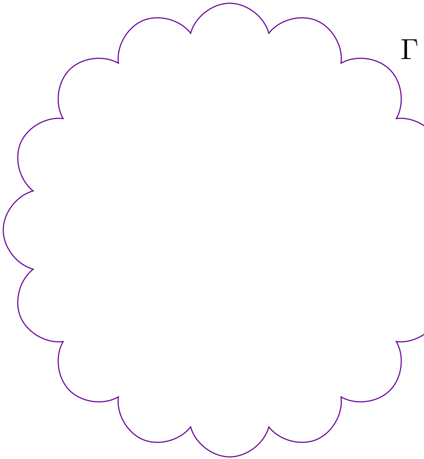

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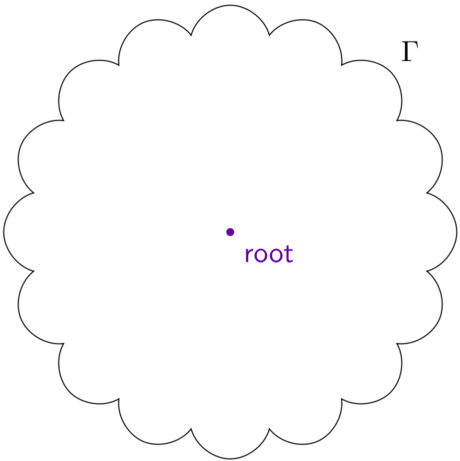

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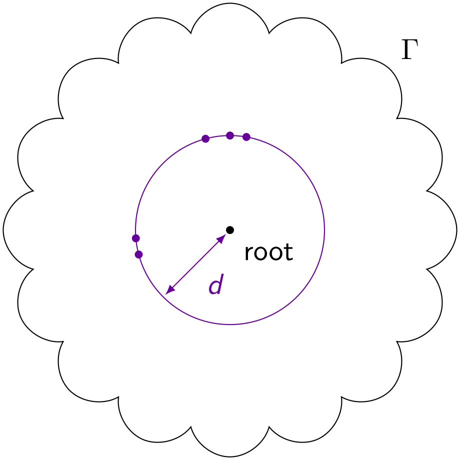

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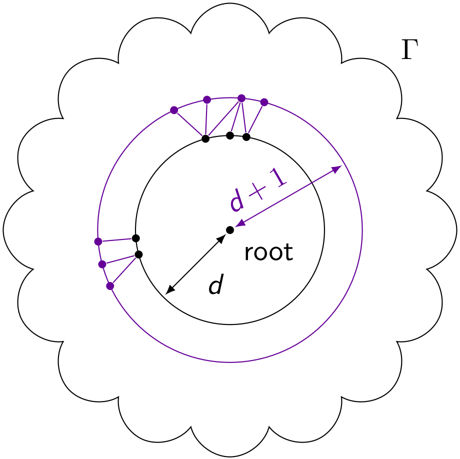

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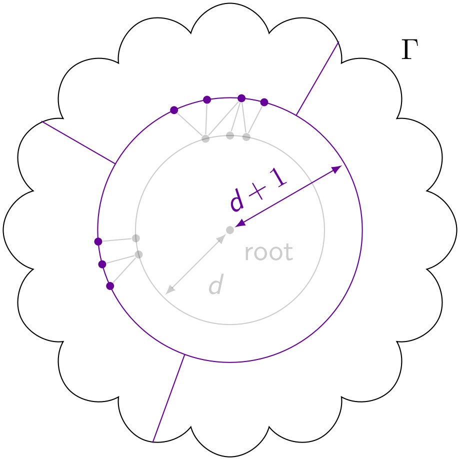

"end-cone"

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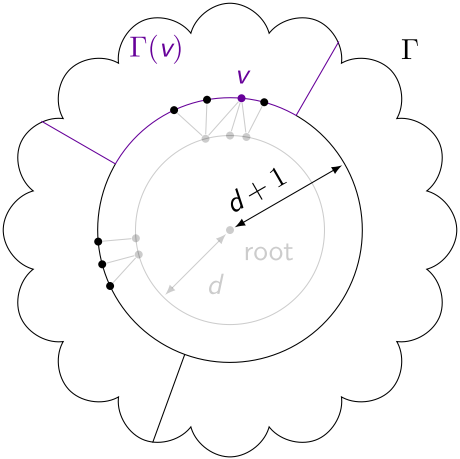

"end-cone" of v

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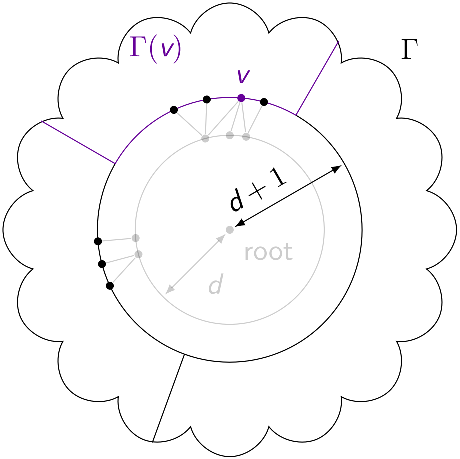

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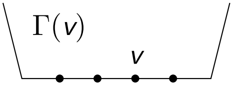

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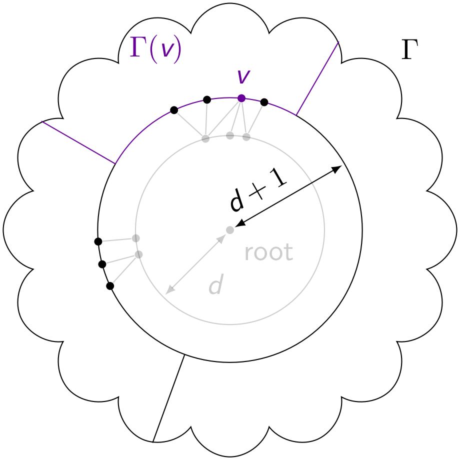

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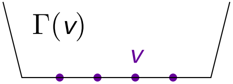

"frontier points"

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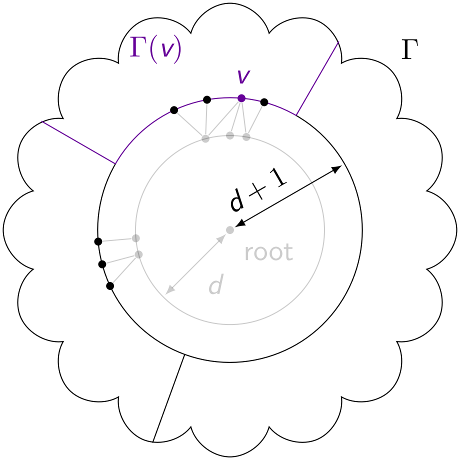

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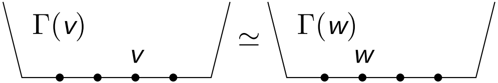

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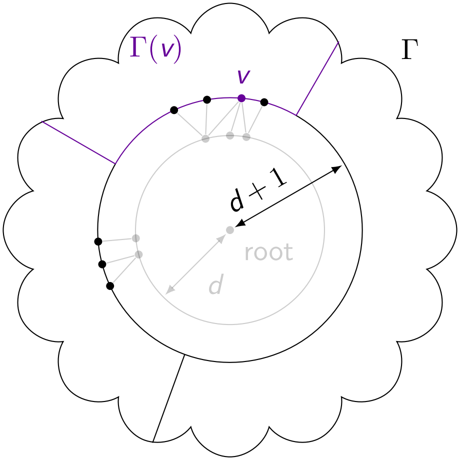

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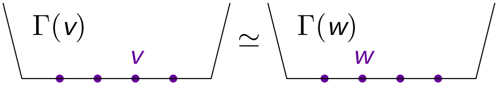

"frontier points"

"end isomorphism":

maps frontier points to frontier points

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nodes: $Q \times \Gamma^*$

edges: $(p, A\gamma) \xrightarrow{a} (q, \beta\gamma)$

if $(p, A, a) \to (q, \beta) \in \delta$

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&\iff \mathscr{L}(\Gamma; u, u) \text{ is deterministic context-free} && \text{(Rodaro, 2024)}
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$\implies$ Γ is tree-like

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"with some regularity condition"

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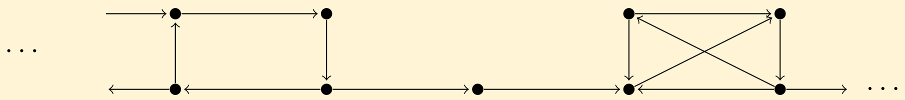

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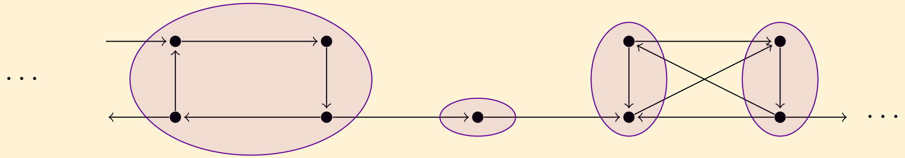

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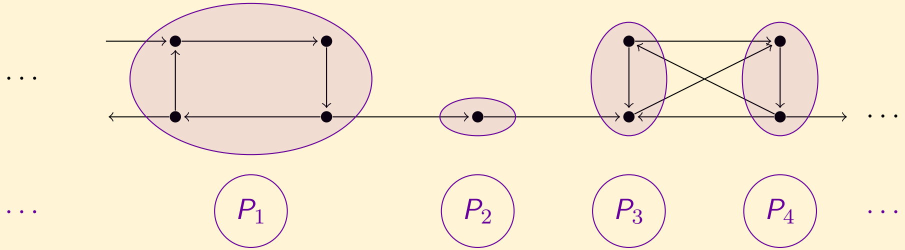

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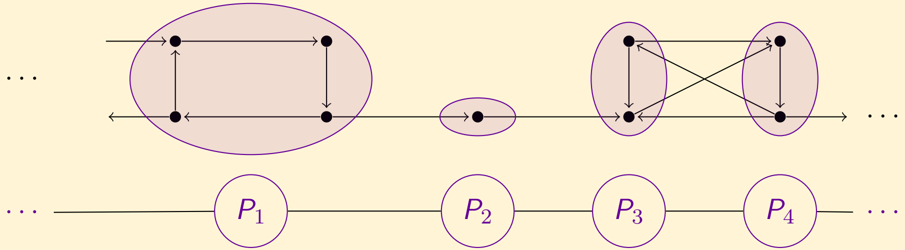

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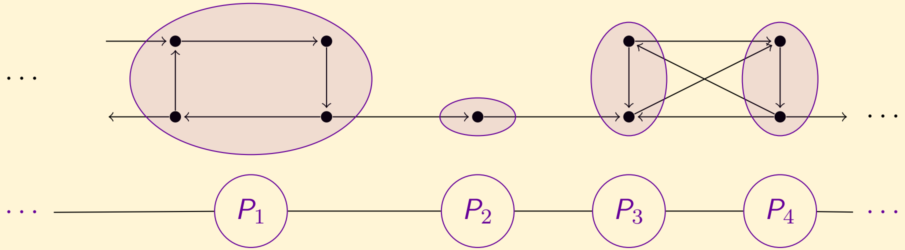

Idea: This tree needs to be "context-free" in some sense!

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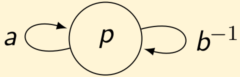

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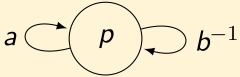
Every state p of a pDFA generates an involutive tree $\Gamma(p)$ via its tree of runs/path space.

Example

p ε

This is the root!



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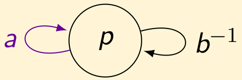

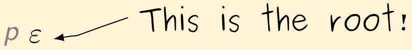

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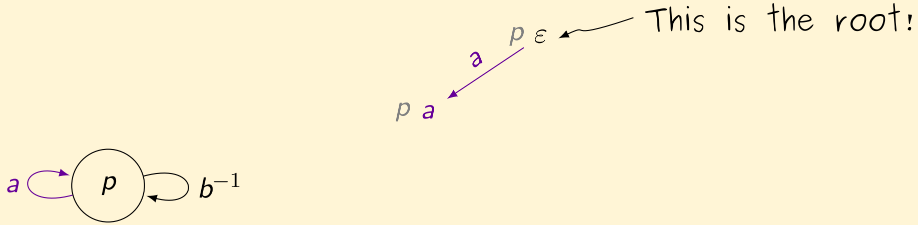

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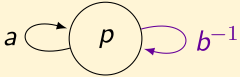

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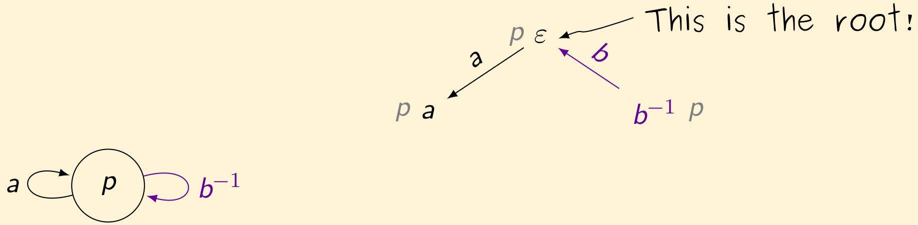

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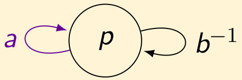

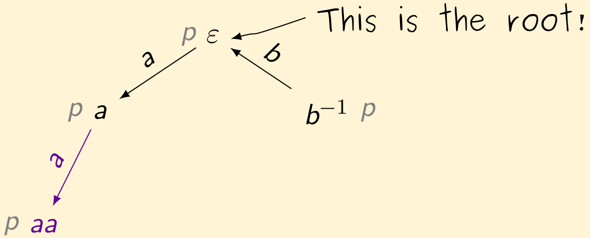

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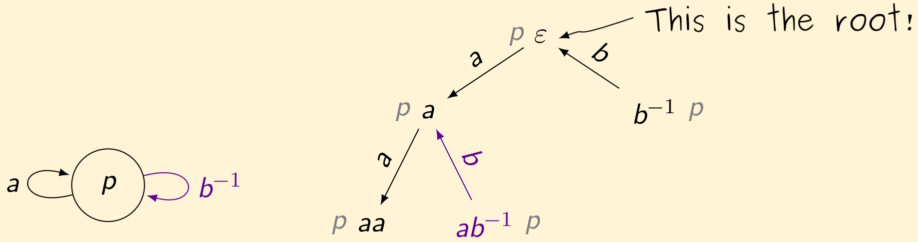

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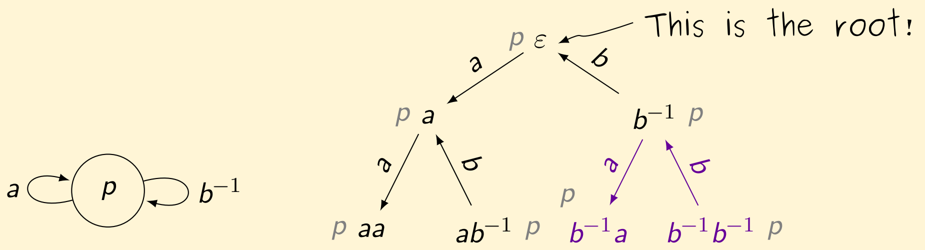

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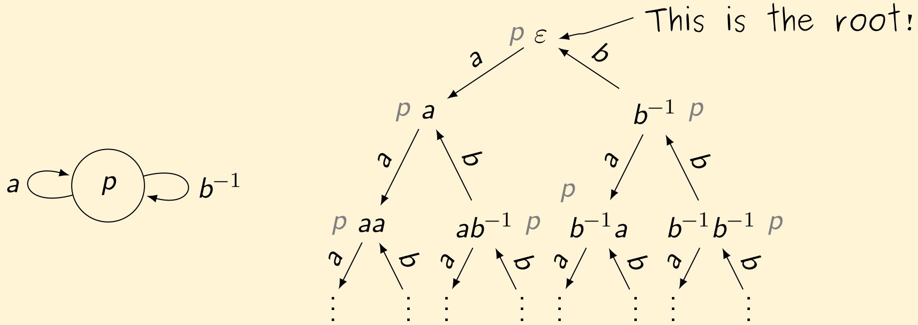

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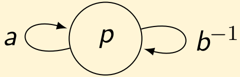

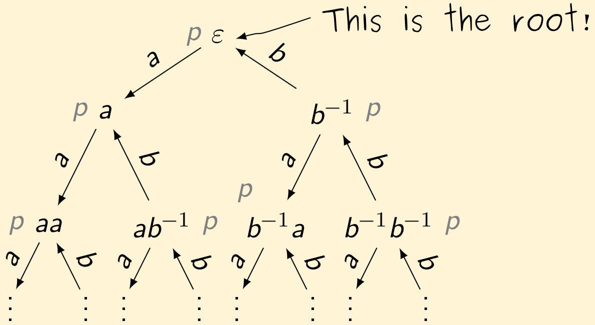

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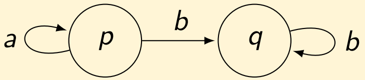

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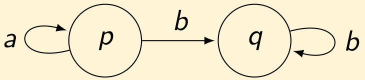

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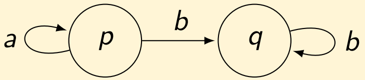

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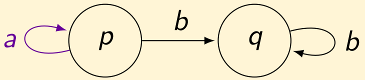

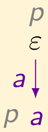

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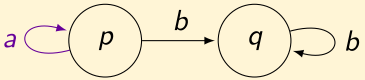

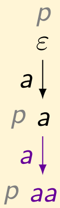

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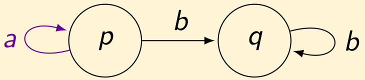

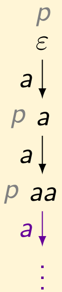

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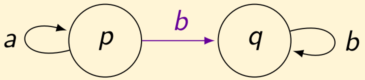

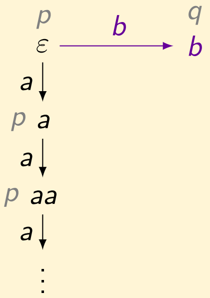

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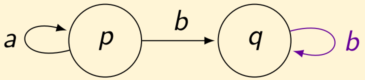

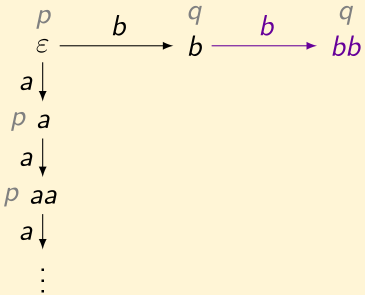

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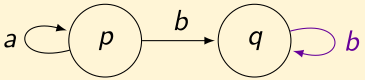

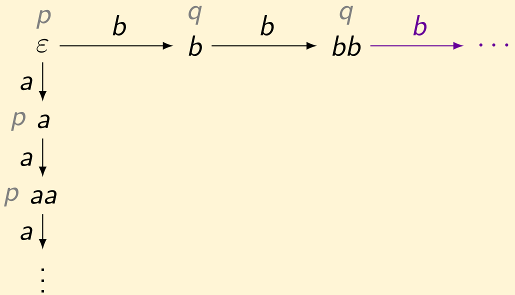

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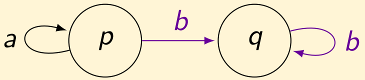

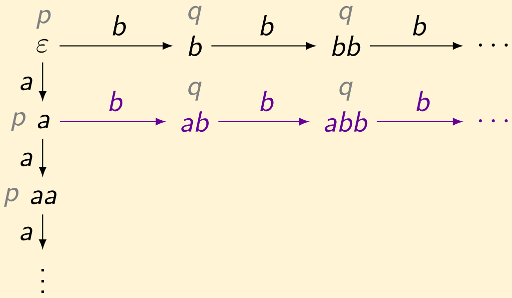

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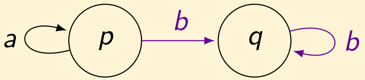

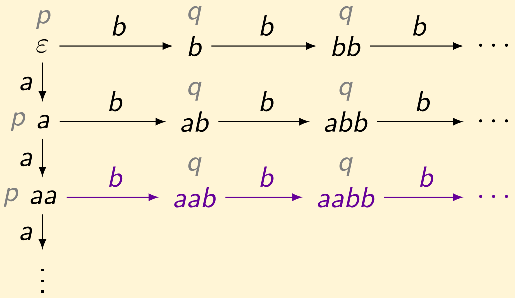

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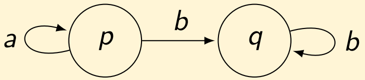

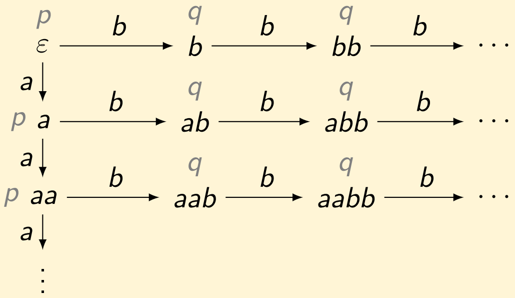

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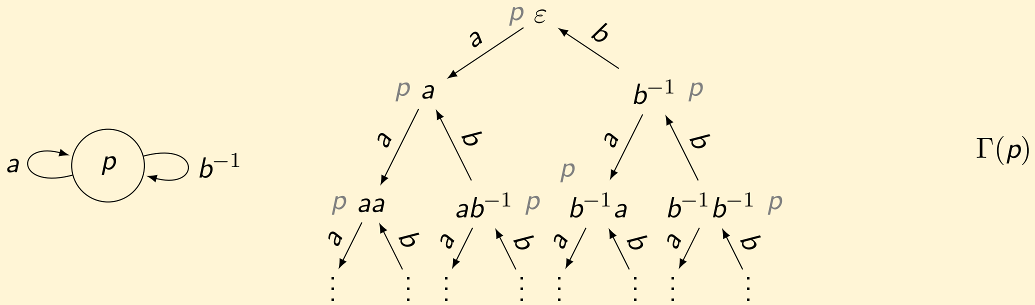

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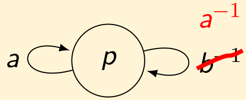

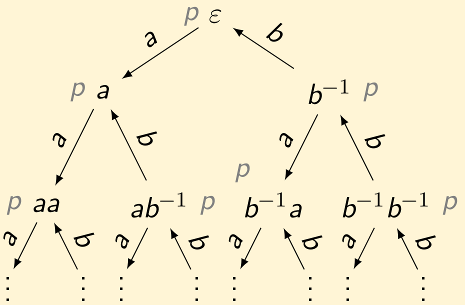

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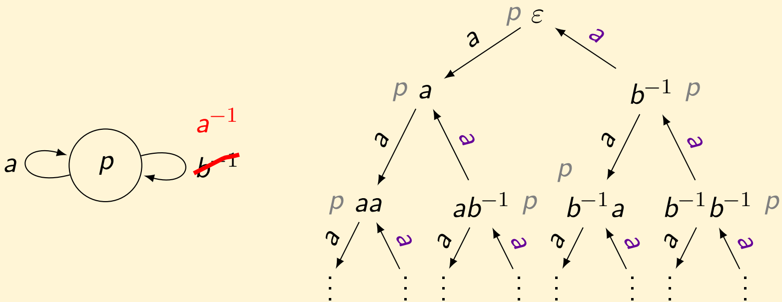

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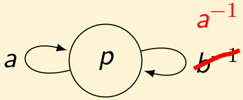

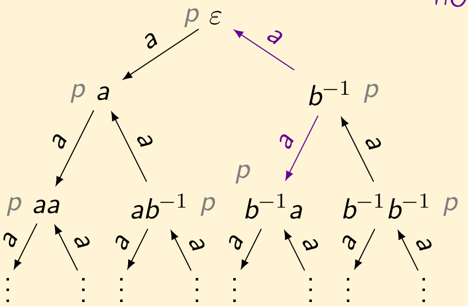

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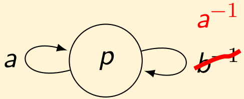

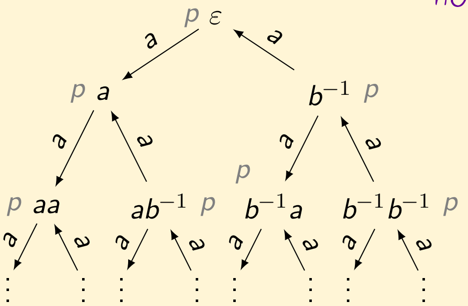

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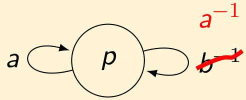

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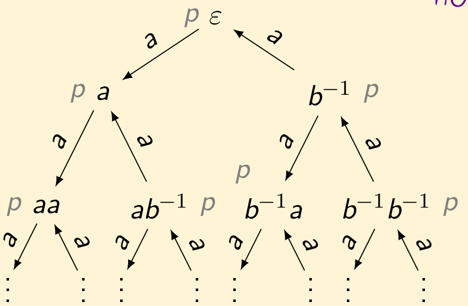

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$\iff$: the pDFA is reduced

Context-Free Trees

Γ: inverse $A^{\pm 1}$-tree, i. e. a tree as an unlabeled, undirected graph

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Theorem (W., arXiv 2026)

Γ *is context-free*

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Γ *is context-free* $\iff$ Γ *arises as* $\Gamma(p)$ *for a state* p *of a reduced pDFA*

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Stephen (1990): Deciding Green's relations $(\mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{J})$ is closely related to the (rooted) isomorphism (homomorphism) problem for the Schützenberger graphs.

The Rooted Isomorphism Problem

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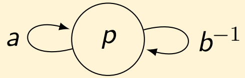

The Rooted Isomorphism Problem

$\mathscr{L}(p)$: words readable from state p in its pDFA

Example

$$p \xrightarrow{a} p, \quad p \xrightarrow{b^{-1}} p$$

$\mathscr{L}(p) = \{a, b^{-1}\}^*$

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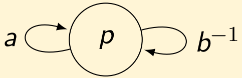

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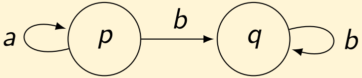

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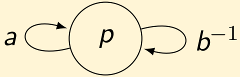

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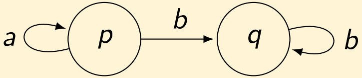

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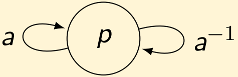
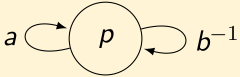


$\mathscr{L}(p) = \{a, a^{-1}\}^*$

The Rooted Isomorphism Problem

$\mathscr{L}(p)$: words readable from state p in its pDFA

Example



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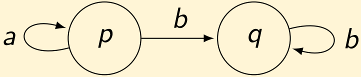

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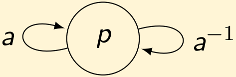

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Fact

Let p and q be states of reduced pDFAs. Then:

$$\Gamma(p) \doteq \Gamma(q) \iff \mathscr{L}(p) = \mathscr{L}(q)$$

isomorphic as **rooted** graphs

The Rooted Isomorphism Problem

$\mathscr{L}(p)$: words readable from state p in its pDFA

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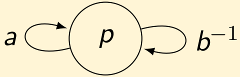

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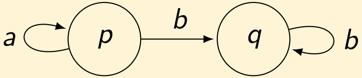

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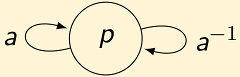

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It is well known that the latter is NL-complete.

The Rooted Isomorphism Problem

$\mathscr{L}(p)$: words readable from state p in its pDFA

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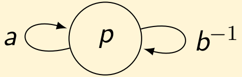

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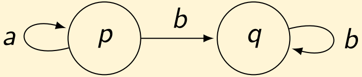

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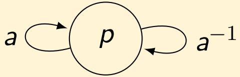

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Let p and q be states of reduced pDFAs. Then:

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Adapting a construction by Cho & Huynh, we may assume every state to be accessible from p and q, respectively (for the lower bound).

The Non-Rooted Case: The Problem

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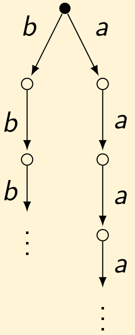
Example

Δ



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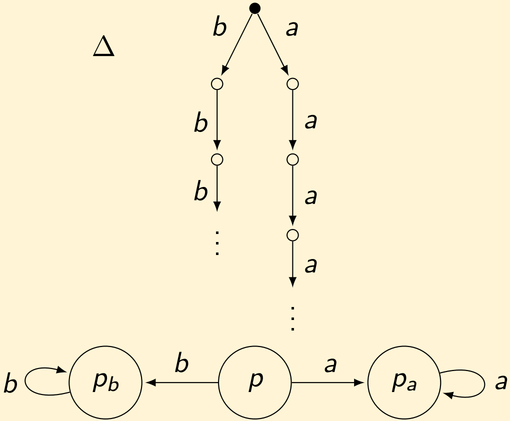

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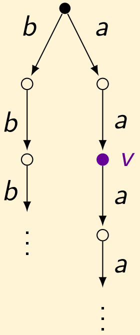

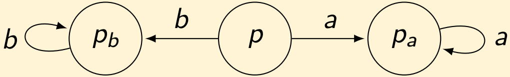

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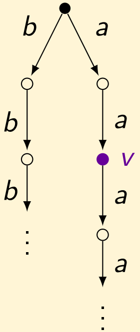
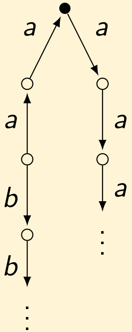
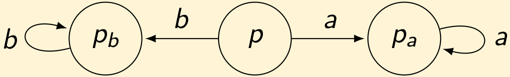


$\Delta(v)$





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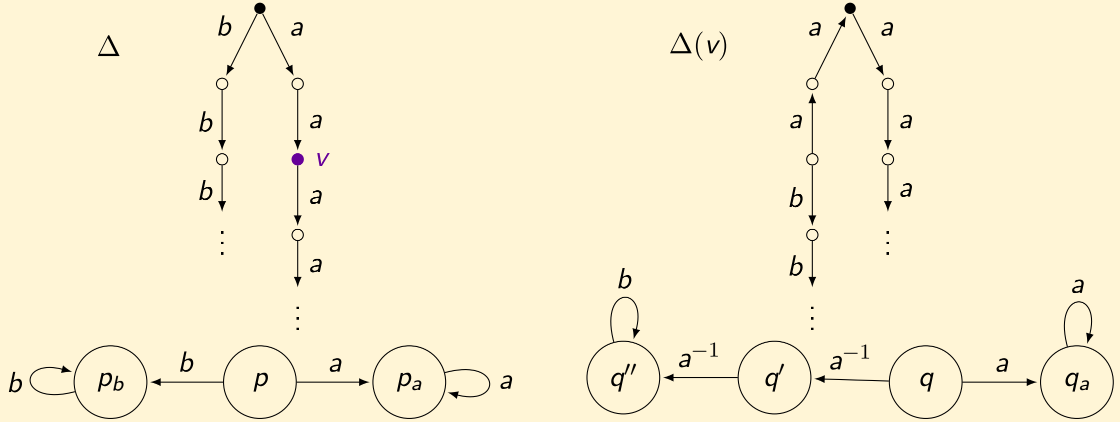

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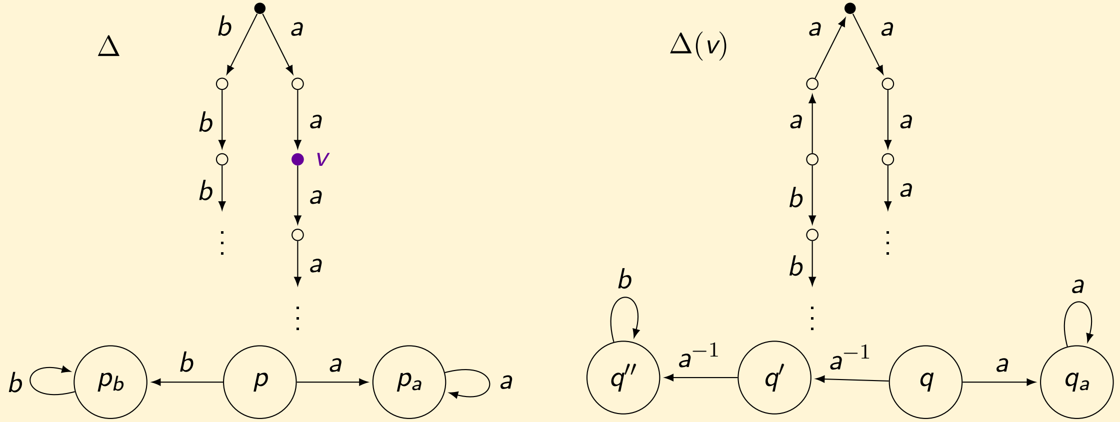

The languages don't seem connected!

The Non-Rooted Isomorphism Problem

Theorem (W., arXiv 2026)

The non-rooted isomorphism problem for inverse context-free trees

Input:	*states* $\check{p}$ *and* $\check{q}$ *of reduced pDFAs*
Question:	*are* $\Gamma(\check{p})$ *and* $\Gamma(\check{q})$ *isomorphic as non-rooted graphs*

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The tricky part is "$\in \mathrm{NL}$".

Proof Idea

Proof Idea

p: state of reduced pDFA $\mathcal{A}$,

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Predicate: $U(p, q)$

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Predicate: $U(p, q) \iff \exists$ node $v \in \Delta$

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Predicate: $U(p, q) \iff \exists$ node $v \in \Delta$ labeled by q

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Predicate: $U(p, q) \iff \exists\,\text{node } v \in \Delta \text{ labeled by } q : \Gamma(p) \doteqdot \Delta(v)$

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$$\implies q = \check{q},\ \Delta(v) = \Delta$$

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$\implies q = \check{q}, \ \Delta(v) = \Delta$, i.e. $U(p, \check{q})$

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Thus: $U(p, q) \iff \big(q = \check{q} \text{ and } \mathscr{L}(p) = \mathscr{L}(\check{q})\big)$

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$\Gamma(p) \doteq \Delta(v) = \Delta = \Gamma(\check{q}) \iff \mathscr{L}(p) = \mathscr{L}(\check{q})$

Thus: $U(p, q) \iff \left(q = \check{q} \text{ and } \mathscr{L}(p) = \mathscr{L}(\check{q})\right)$ or "v is not the root"

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$\mathcal{A}$ p ε

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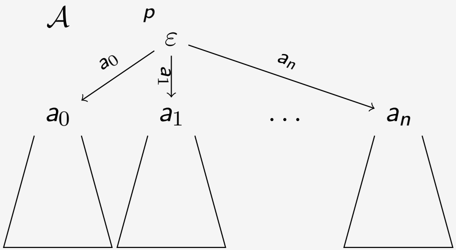

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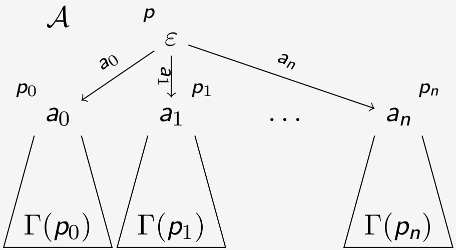

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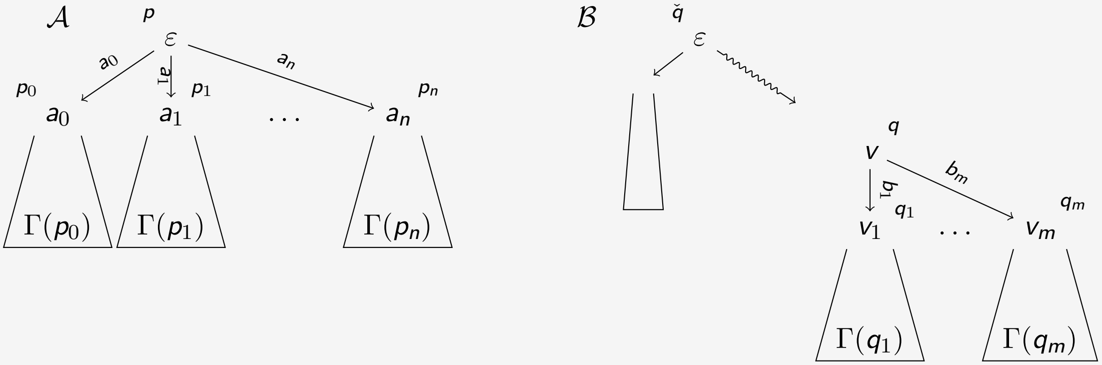

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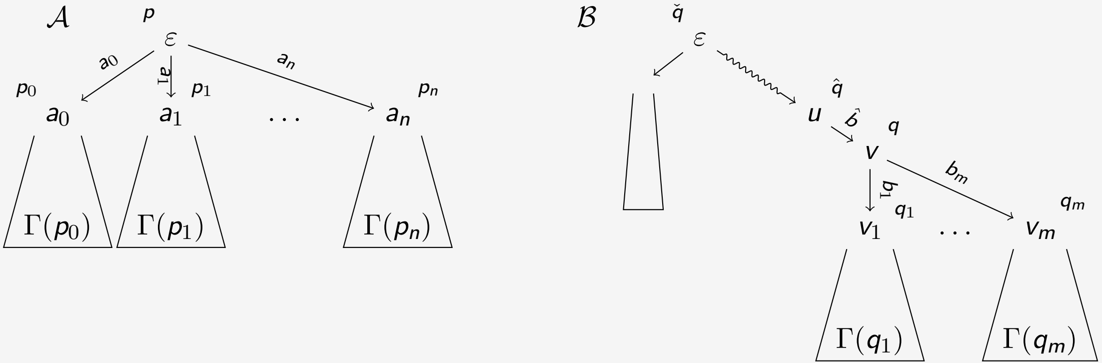

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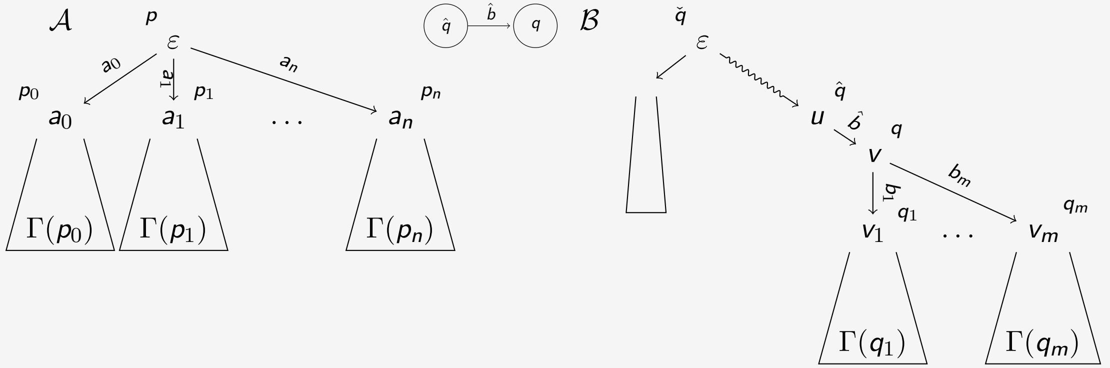

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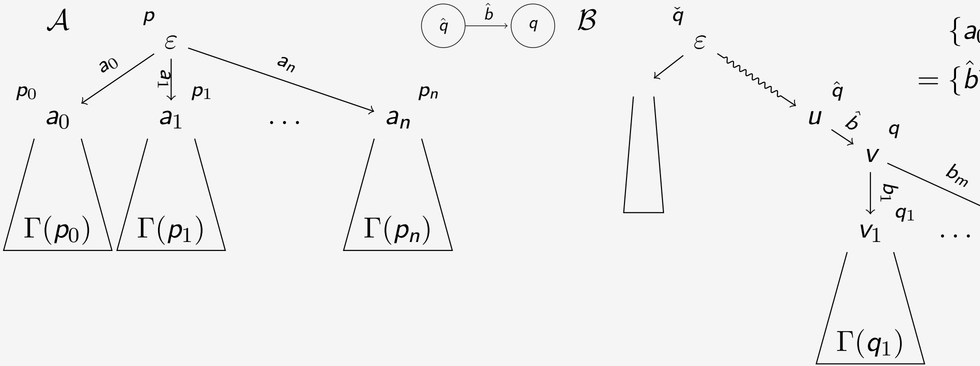

$$\{a_0, \dots, a_n\}$$
$$= \{\hat{b}^{-1}, b_1, \dots, b_m\}$$

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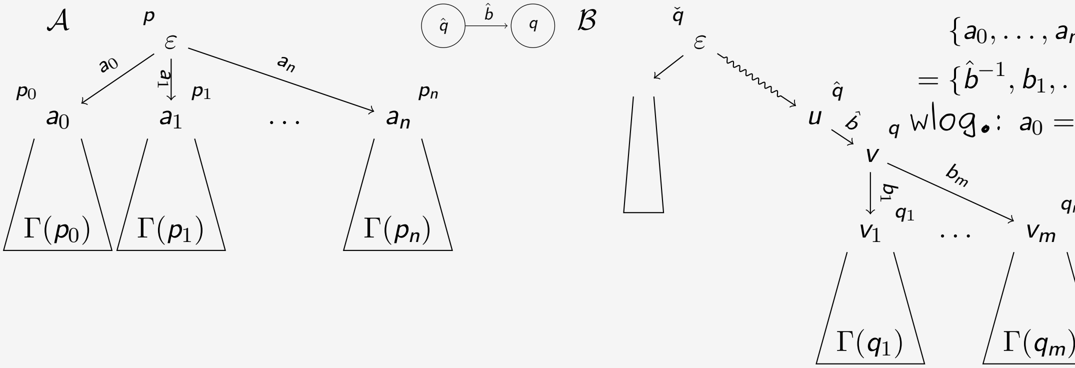

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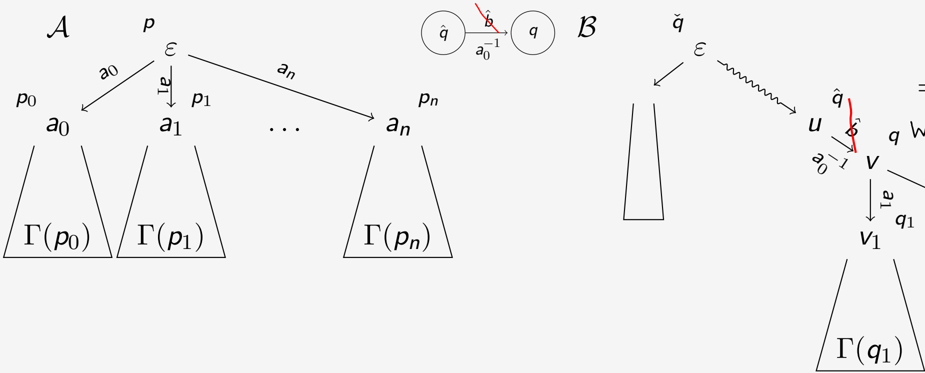

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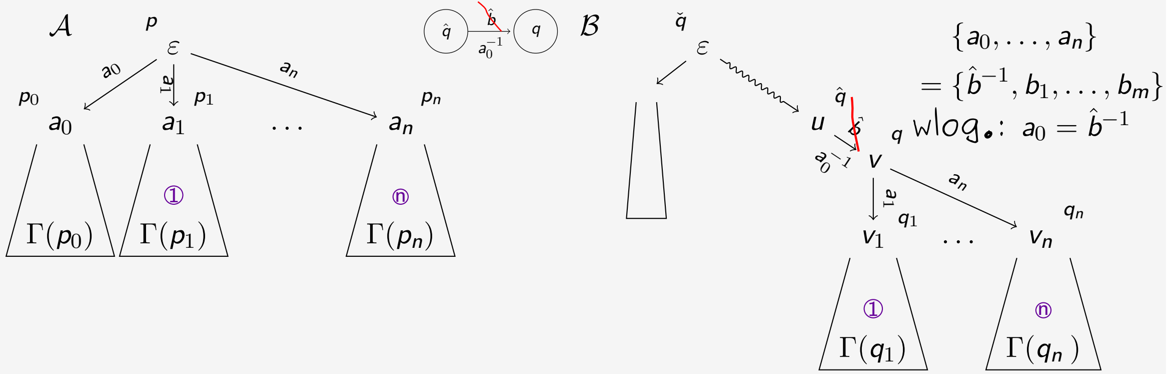

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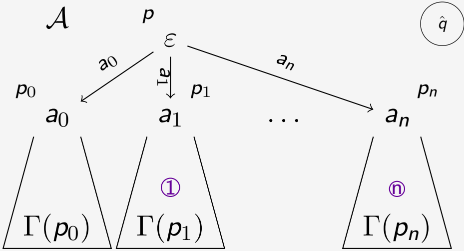

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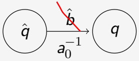

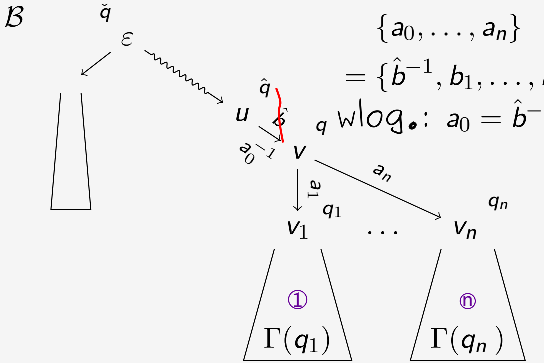

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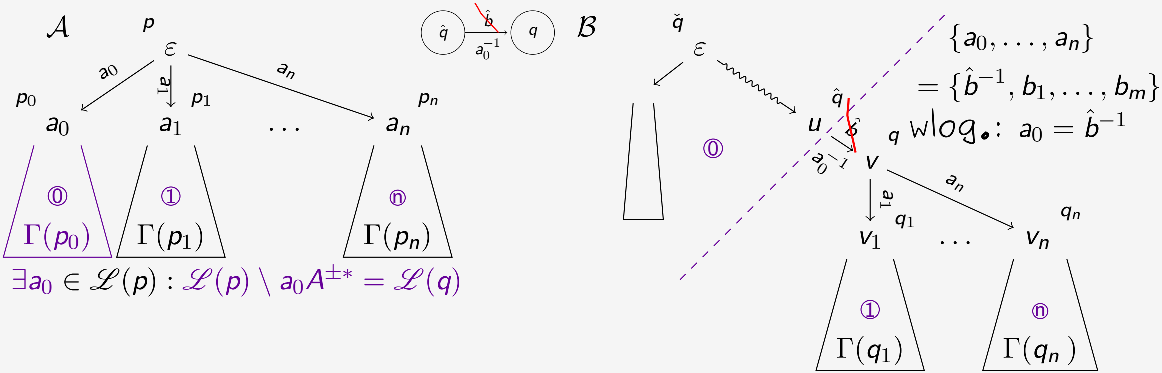

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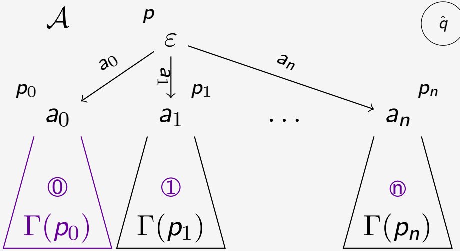

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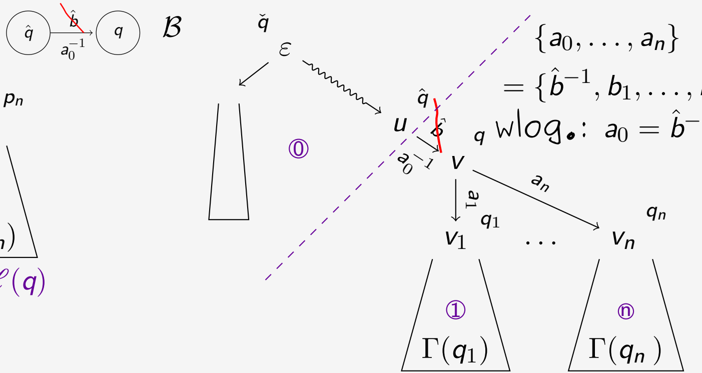
$\& \; \exists \hat{q} : \hat{q} \xrightarrow{a_0^{-1}} q \in \mathcal{B}$



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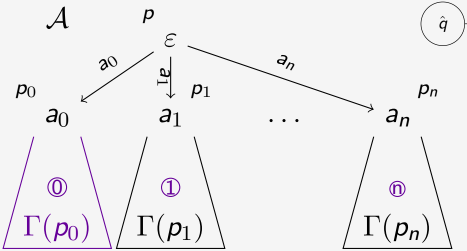

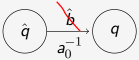

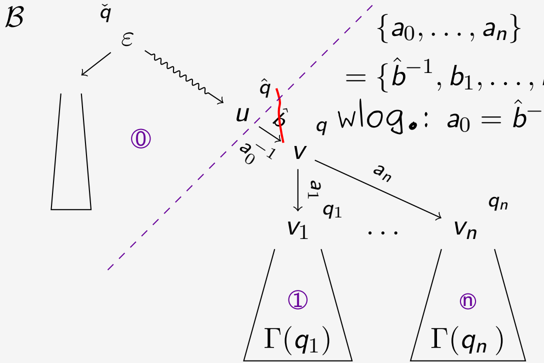

$\{a_0, \dots, a_n\}$

$= \{\hat{b}^{-1}, b_1, \dots, b_m\}$

wlog.: $a_0 = \hat{b}^{-1}$

$\exists a_0 \in \mathscr{L}(p) : \mathscr{L}(p) \setminus a_0 A^{\pm *} = \mathscr{L}(q)$

$\& \ \exists \hat{q} : \hat{q} \xrightarrow{a_0^{-1}} q \in \mathcal{B}$ and

Proof Idea

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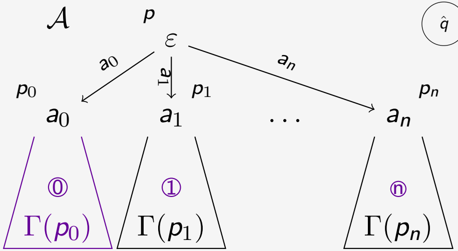

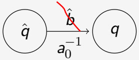

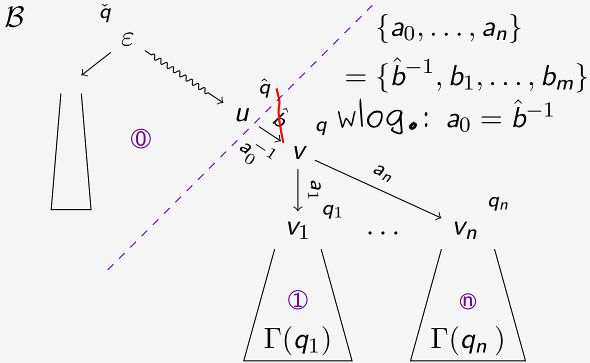

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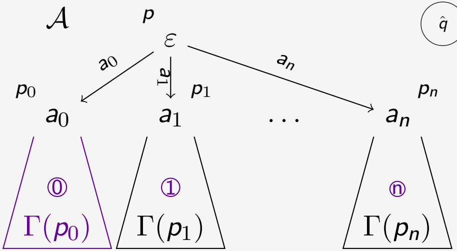

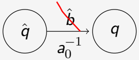

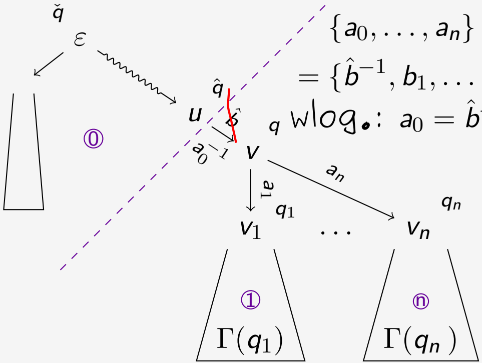

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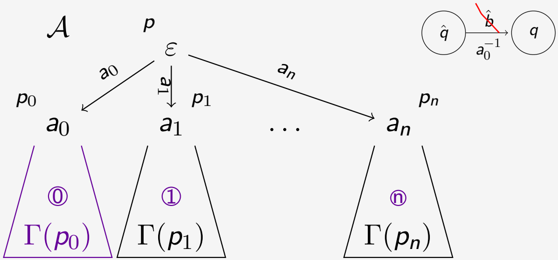

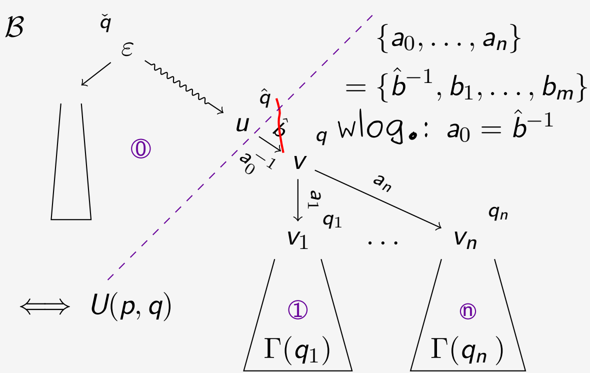

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$\iff U(p, q)$

Proof Idea

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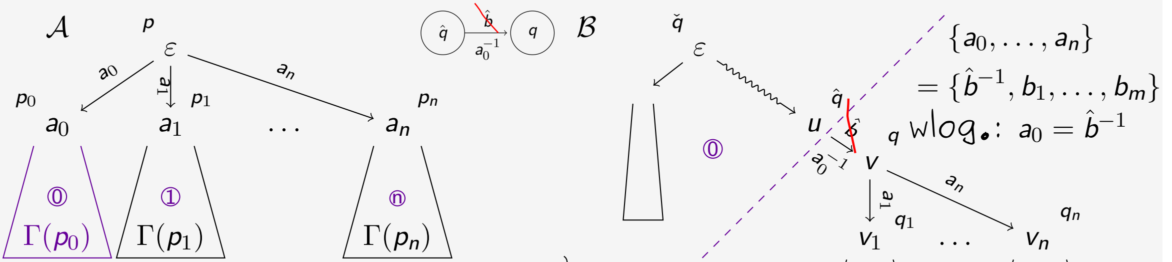

$\left.\begin{array}{l}\exists a_0 \in \mathscr{L}(p) : \mathscr{L}(p) \setminus a_0 A^{\pm *} = \mathscr{L}(q) \backslash b_0 A^{\pm *} \\ \& \ \exists \hat{q} : \hat{q} \xrightarrow{a_0^{-1}} q \in \mathcal{B} \text{ and } U(\underbrace{p \cdot a_0}_{p_0}, \hat{q}, a_0^{-1})\end{array}\right\} \iff U(p, q, b_0)$

Summary

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$$U(p, q, b_0) \iff$$

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or

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and $\exists \hat{q}$:

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NL−Algorithm: Guess proof for $U(\check{p}, q)$ for some q nondeterministically.

Outlook

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1. Extend this to general inverse context-free graphs.

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1. Extend this to general inverse context-free graphs. PSPACE?
2. Compute the generating automaton from an inverse semigroup/group presentation.

Thank you!