Context-Free Trees

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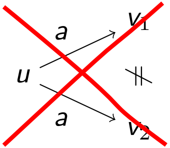
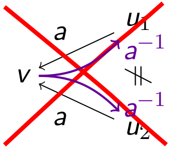
Preliminaries

- involutive alphabet: alphabet A with involution $\cdot^{-1}: A \to A$ (i.e. $(a^{-1})^{-1} = a$ for all $a \in A$)
  $A^{-1} = \{a^{-1} \mid a \in A\}$: disjoint copy $\quad A^{\pm 1} = A \uplus A^{-1}$: involutive alphabet
- A-graph: $\Gamma = (V, E)$ with $E \subseteq V \times A \times V$ (i.e. directed, A-edge-labeled)
  Today: usually with a root
- involutive A-graph Γ: A involutive alphabet and $u \xrightarrow{a} v$ in Γ $\iff$ $u \xleftarrow{a^{-1}} v$ in Γ
- deterministic A-graph Γ:



in Γ

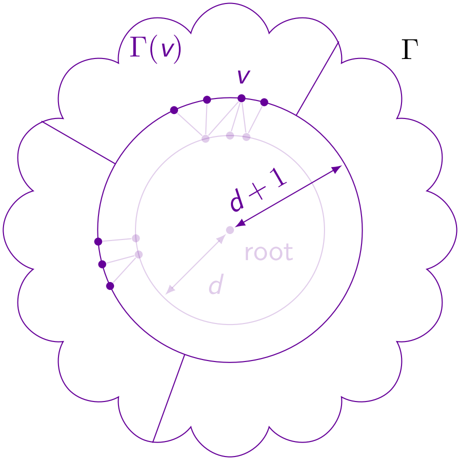
- co-deterministic A-graph Γ:



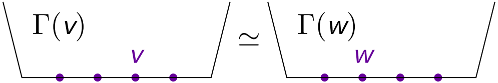
in Γ

- inverse A-graph: involutive + strongly connected +
  deterministic + co-deterministic

End-Cones and End-Isomorphisms



"end-cone" of v



"frontier points"

"end isomorphism":

maps frontier points to frontier points

Context-Free Graphs

Definition (Muller, Schupp; 1985)

A context-free graph is a

- rooted,
- connected and
- involutive
- *A*-graph for
- finite alphabet *A* with
- uniformly bounded degree and
- finitely many end-isomorphism classes.

Corollary (Muller, Schupp; 1985)

Being context-free is independent of the choice of the root.

Theorem (Muller, Schupp; 1985)

Γ *is context-free*

$\iff \Gamma = \Delta^{\pm 1}$ *for a configuration graph of a PDA*

nodes: $Q \times \Gamma^*$

edges: $(p, A\gamma) \xrightarrow{a} (q, \beta\gamma)$

if $(p, A, a) \to (q, \beta) \in \delta$

Context-Free Graphs in the Wild

Context-free graphs appear naturally as

- Cayley graphs of context-free groups and
- Schützenberger graphs of some finitely presented inverse monoids.

These are inverse graphs!

Some details on ε-transitions and $\perp$ omitted

Γ: inverse $A^{\pm 1}$-graph with root u Then:

Γ is context-free $\iff$ Γ has finitely many end-isomorphism classes (definition)

$\iff$ Γ is the configuration graph of some PDA (Muller, Schupp, 1990)

$\iff$ $\mathscr{L}(\Gamma; u, u)$ is deterministic context-free (Rodaro, 2024)

$\iff$ Γ is tree-like

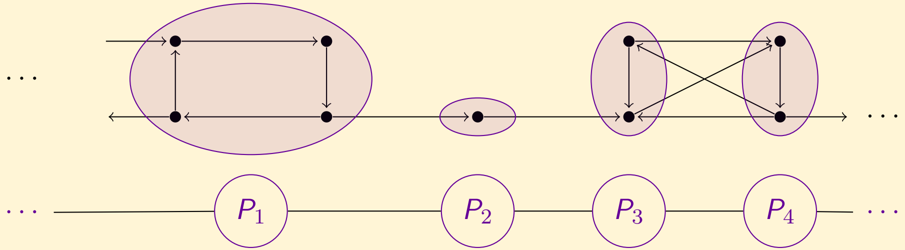
labels of cycles at u

"with some regularity condition"

Tree-Like Graphs

A directed graph is tree-like
$\iff$ it is quasi-isometric to a tree
$\iff$ it has a strong tree decomposition

Example

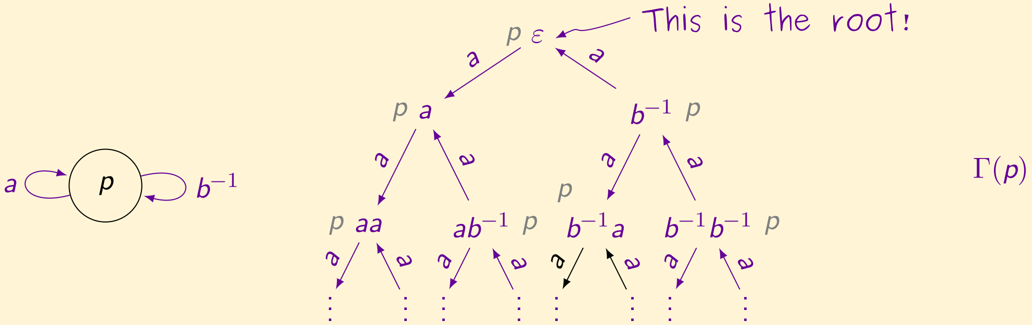


Idea: This tree needs to be “context-free” in some sense!

pDFA: partial deterministic finite automaton over $A^{\pm 1}$

Every state p of a pDFA generates an involutive tree $\Gamma(p)$ via its tree of runs/path space.

Example



pDFAs Generating Trees

pDFA: partial deterministic finite automaton over $A^{\pm 1}$

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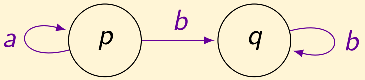
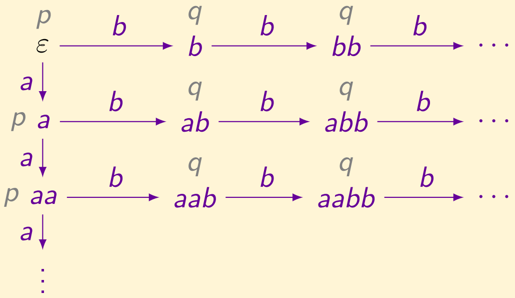
Example



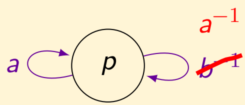


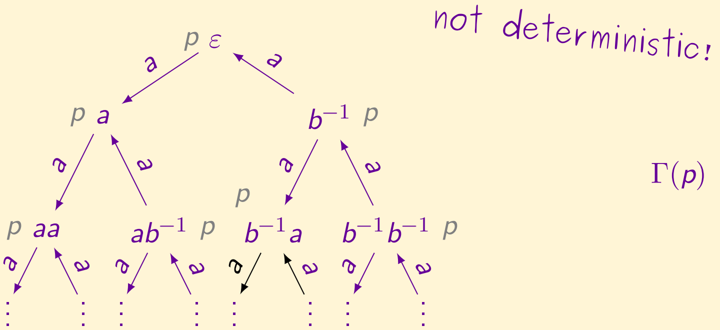
$\Gamma(p)$

pDFA: partial deterministic finite automaton over $A^{\pm 1}$

Every state p of a pDFA generates an involutive tree $\Gamma(p)$ via its tree of runs/path space.

Example





$\Gamma(p)$ is inverse and deterministic $\iff$ aa^{-1} cannot be read for any $a \in A^{\pm 1}$

$\iff$: the pDFA is reduced

Context-Free Trees

Γ: inverse $A^{\pm 1}$-tree, i. e. a tree as an unlabeled, undirected graph

Theorem (W., arXiv 2026)

Γ is context-free $\iff$ Γ arises as $\Gamma(p)$ for a state p of a reduced pDFA

Context-free trees

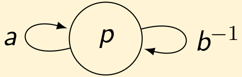
- arise naturally as Schützenberger graphs of
  - free inverse monoids (Munn trees) and as Caley graphs of free groups.
  - inverse monoids with idempotent relators.
- hopefully serve as a stepping stone for general context-free graphs.

Stephen (1990): Deciding Green's relations $(\mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{J})$ is closely related to the (rooted) isomorphism (homomorphism) problem for the Schützenberger graphs.

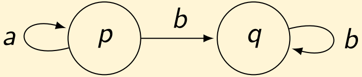
The Rooted Isomorphism Problem

$\mathscr{L}(p)$: words readable from state p in its pDFA

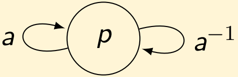
Example



$\mathscr{L}(p) = \{a, b^{-1}\}^*$



$\mathscr{L}(p) = a^* b^*$



$\mathscr{L}(p) = \{a, a^{-1}\}^*$

Fact

Let p and q be states of reduced pDFAs. Then:

$$\Gamma(p) \doteqdot \Gamma(q) \iff \mathscr{L}(p) = \mathscr{L}(q)$$

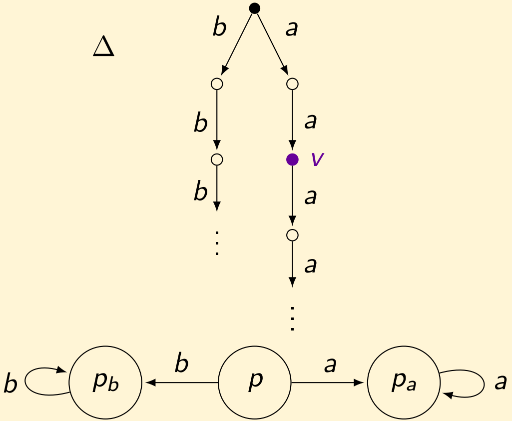
isomorphic as **rooted** graphs

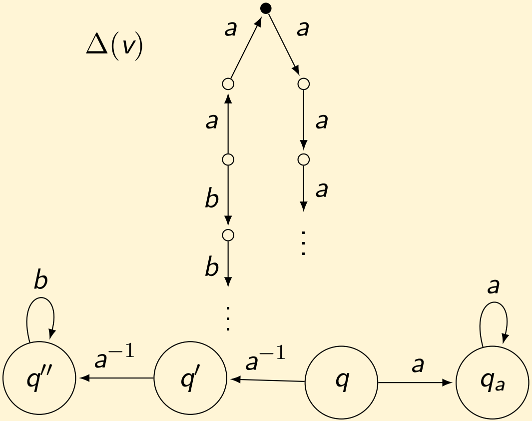
It is well known that the latter is NL-complete.

Adapting a construction by Cho & Huynh, we may assume every state to be accessible from p and q, respectively (for the lower bound).

The Non-Rooted Case: The Problem

Example





The languages don't seem connected!

The Non-Rooted Isomorphism Problem

Theorem (W., arXiv 2026)

The non-rooted isomorphism problem for inverse context-free trees

Input:	*states* $\check{p}$ *and* $\check{q}$ *of reduced pDFAs*
Question:	*are* $\Gamma(\check{p})$ *and* $\Gamma(\check{q})$ *isomorphic as non-rooted graphs*

is in NL*-complete.*

The tricky part is "$\in$ NL".

Proof Idea

p: state of reduced pDFA $\mathcal{A}$, $\check{q}$: state of a reduced pDFA $\mathcal{B}$, $\Delta = \Gamma(\check{q})$

Predicate: $U(p, q, b_0) \iff \exists\,$node $v \in \Delta$ labeled by $q : \Gamma(p) \doteq \Delta(v) \setminus v\, b_0 A^{\pm *}$

Case: v is the root

$\implies q = \check{q},\ \Delta(v) = \Delta$, i.e. $U(p, \check{q})$

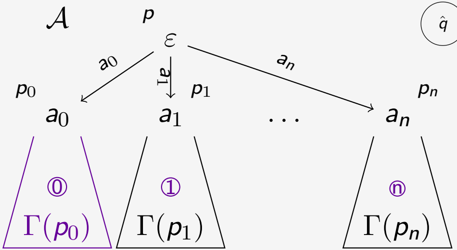
$\Gamma(p) \doteq \Delta(v) = \Delta = \Gamma(\check{q}) \iff \mathscr{L}(p) = \mathscr{L}(\check{q})$

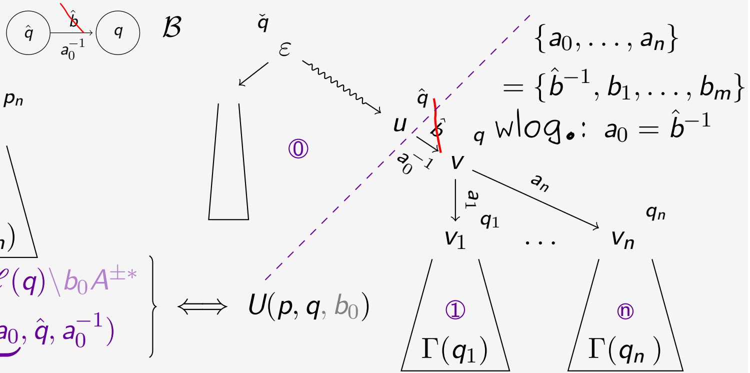
Thus: $U(p, q) \iff \big(q = \check{q}$ and $\mathscr{L}(p) = \mathscr{L}(\check{q})\big)$ or "v is not the root"

p: state of reduced pDFA $\mathcal{A}$, $\check{q}$: state of a reduced pDFA $\mathcal{B}$, $\Delta = \Gamma(\check{q})$

Predicate: $U(p, q, b_0) \iff \exists$ node $v \in \Delta$ labeled by $q : \Gamma(p) \doteq \Delta(v) \setminus v\, b_0 A^{\pm *}$

Case: v is not the root





$\exists a_0 \in \mathscr{L}(p) : \mathscr{L}(p) \setminus a_0 A^{\pm *} = \mathscr{L}(q) \setminus b_0 A^{\pm *}$

$\& \ \exists \hat{q} : \hat{q} \xrightarrow{a_0^{-1}} q \in \mathcal{B}$ and $U(\underbrace{p \cdot a_0}_{p_0}, \hat{q}, a_0^{-1})$

$\iff U(p, q, b_0)$

Summary

p: state of reduced pDFA $\mathcal{A}$, $\check{q}$: state of a reduced pDFA $\mathcal{B}$, $\Delta = \Gamma(\check{q})$

Predicate: $U(p, q, b_0) \iff \exists$ node $v \in \Delta$ labeled by $q : \Gamma(p) \doteq \Delta(v) \backslash v\, b_0 A^{\pm *}$

$$U(p, q, b_0) \iff \begin{array}{rl} & q = \check{q} \;\&\; \mathscr{L}(p) = \mathscr{L}(\check{q}) \backslash b_0 A^{\pm *} \\ \text{or} & \exists a_0 \in \mathscr{L}(p) : \mathscr{L}(p) \setminus a_0 A^{\pm *} = \mathscr{L}(q) \backslash b_0 A^{\pm *} \\ & \quad \text{and } \exists \hat{q} : \hat{q} \xrightarrow{a_0^{-1}} q \in \mathcal{B} \;\&\; U(p \cdot a_0, \hat{q}, a_0^{-1}) \end{array}$$

NL-Algorithm: Guess proof for $U(\check{p}, q)$ for some q nondeterministically.

Outlook

1. Extend this to general inverse context-free graphs. PSPACE?
2. Compute the generating automaton from an inverse semigroup/group presentation.

Thank you!