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Inverse Semigroups and the Isomorphism Problem for Context-Free Trees

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Department of Mathematics
University of Manchester

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16 September 2026

Preliminaries

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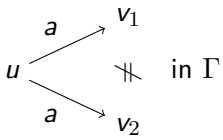

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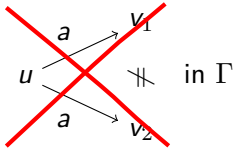

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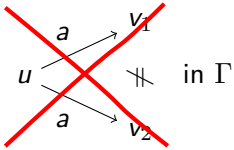

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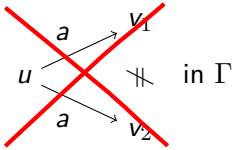

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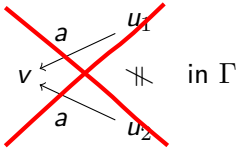

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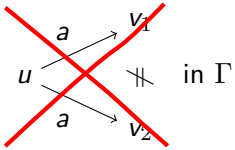

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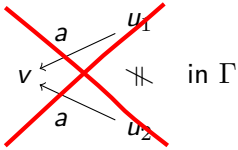

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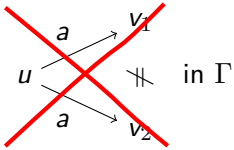

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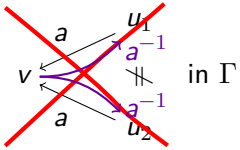

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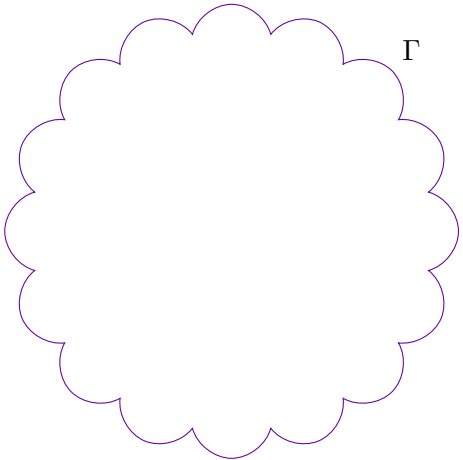

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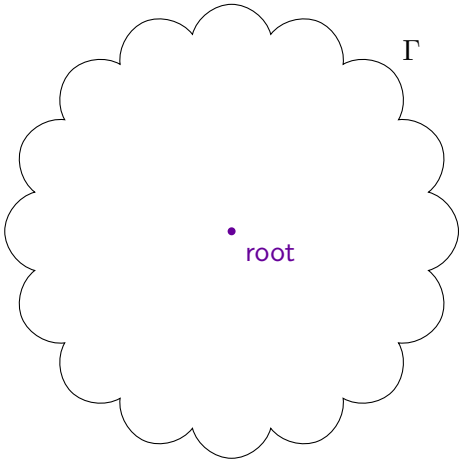

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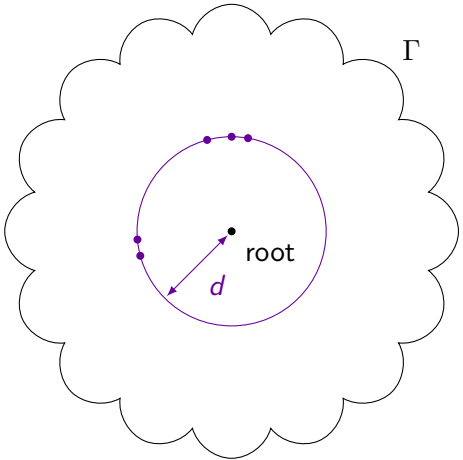

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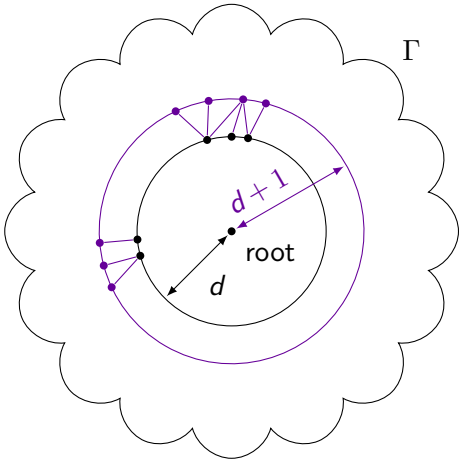

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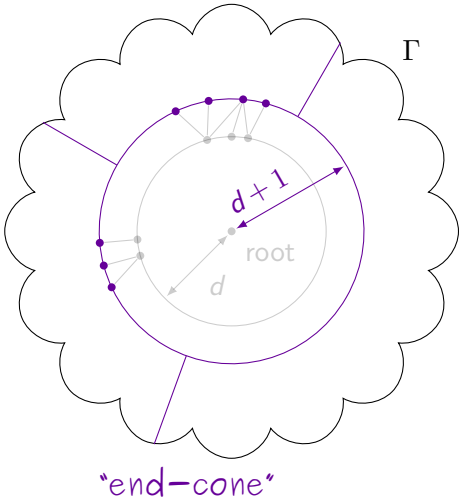

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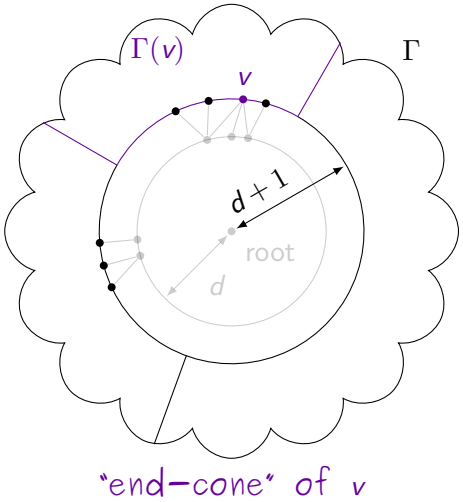

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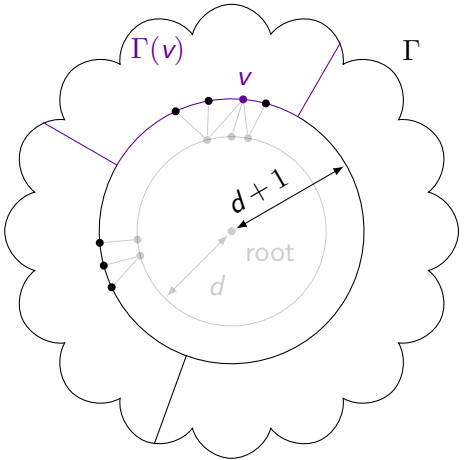

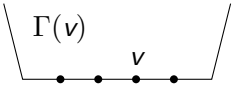

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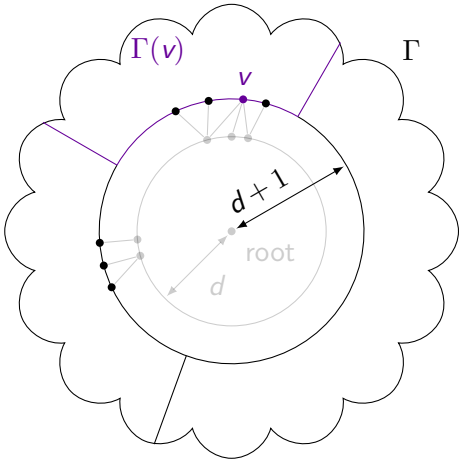

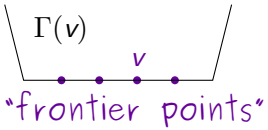

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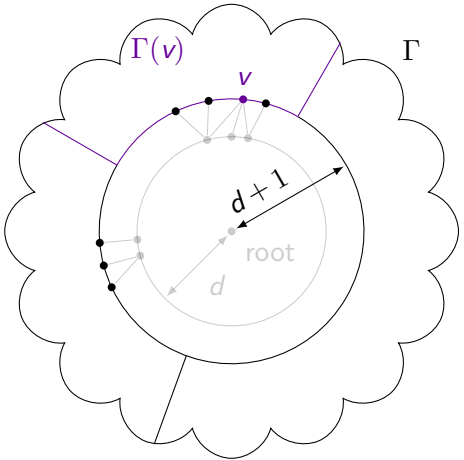

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"frontier points"

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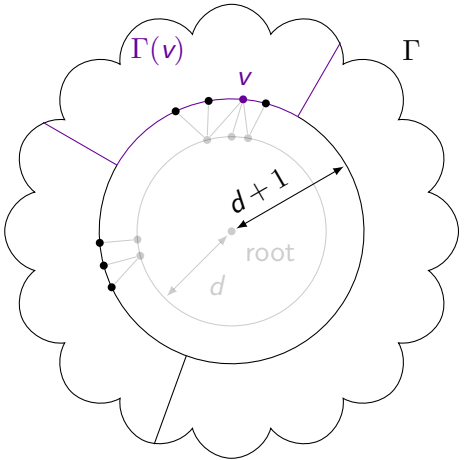

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"end isomorphism":
maps frontier points to frontier points

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Theorem (Muller, Schupp; 1985)

Γ *is context-free*

$\iff \Gamma = \Delta^{\pm 1}$ *for a configuration graph of a PDA*

Context-Free Graphs in the Wild

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These are inverse graphs!

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Γ is context-free $\iff$ Γ has finitely many end-isomorphism classes (definition)

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Γ: inverse $A^{\pm 1}$-graph with root u Then:

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$$\iff \Gamma \text{ is the configuration graph of some PDA} \quad \text{(Muller, Schupp, 1990)}$$
$$\iff \mathscr{L}(\Gamma; u, u) \text{ is deterministic context-free} \quad \text{(Rodaro, 2024)}$$

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$$\iff \mathscr{L}(\Gamma; u, u) \text{ is deterministic context-free} \quad \text{(Rodaro, 2024)}$$
$$\implies \Gamma \text{ is tree-like}$$

$\mathscr{L}(\Gamma; u, u)$: labels of cycles at u

Inverse Context-Free Graphs

Some details on ε-transitions and $\bot$ omitted

Γ: inverse $A^{\pm 1}$-graph with root u Then:

Γ is context-free $\iff$ Γ has finitely many end-isomorphism classes (definition)

$\iff$ Γ is the configuration graph of some PDA (Muller, Schupp, 1990)

$\iff$ $\mathscr{L}(\Gamma; u, u)$ is deterministic context-free (Rodaro, 2024)

$\iff$ Γ is tree-like

labels of cycles at u

"with some regularity condition"

Tree-Like Graphs

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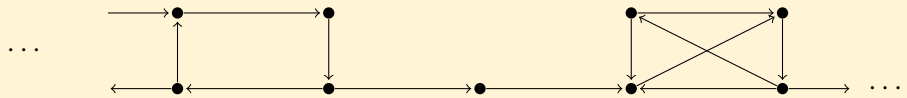

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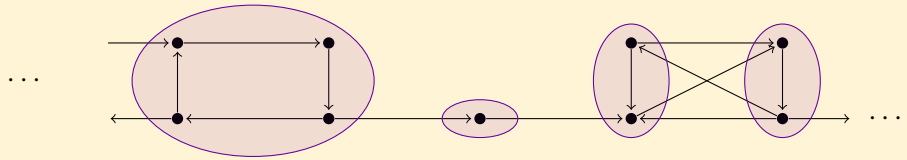

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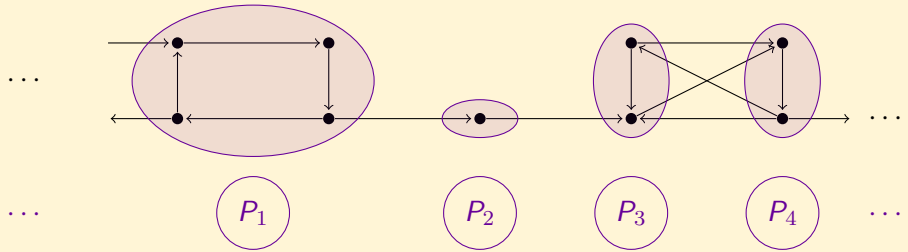

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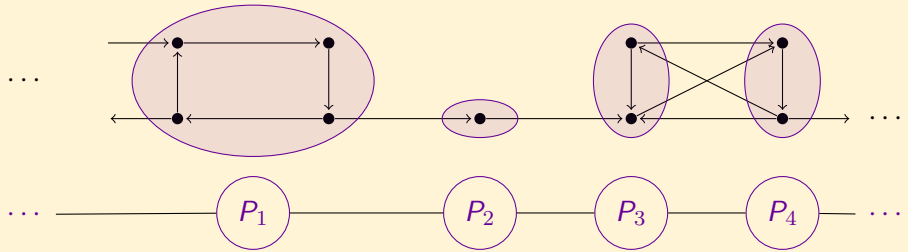

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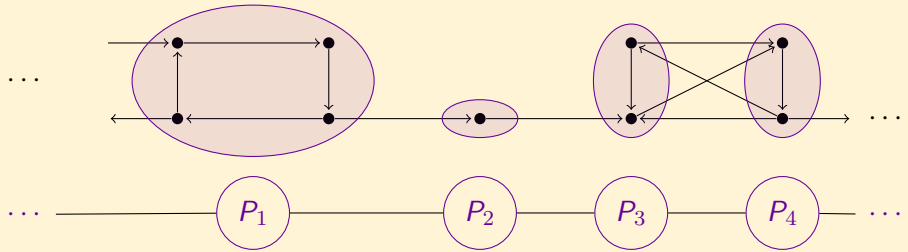

Idea: This tree needs to be "context-free" in some sense!

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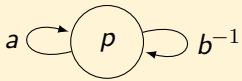

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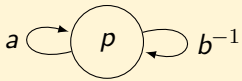
$p\ \varepsilon$ ← This is the root!



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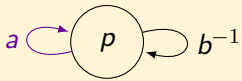

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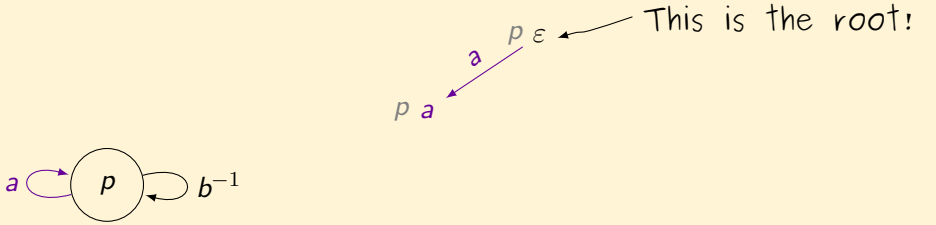

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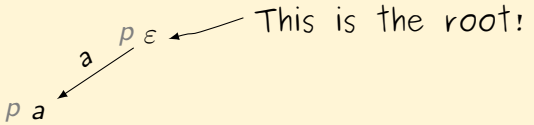

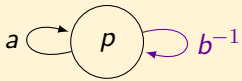

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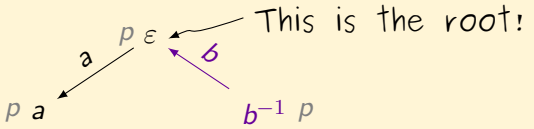

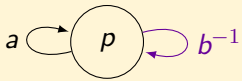

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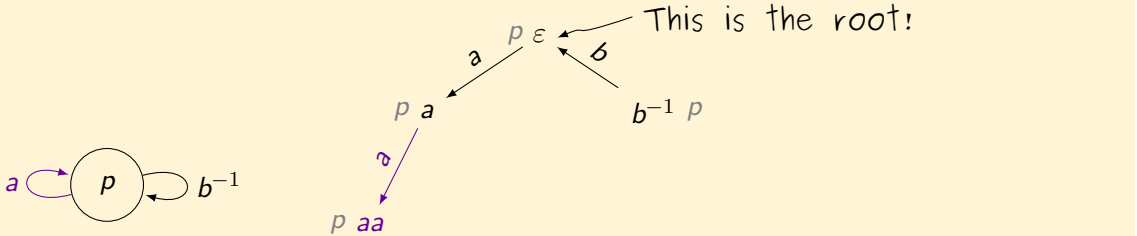

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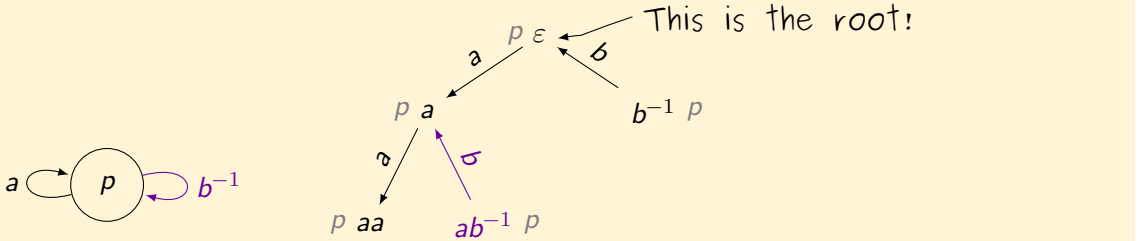

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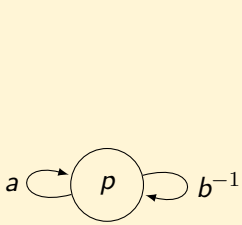

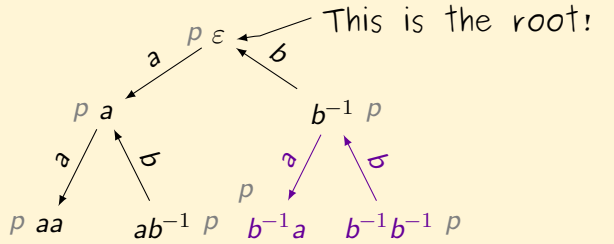

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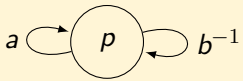

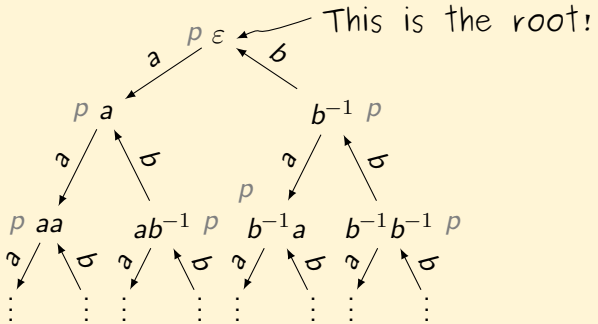

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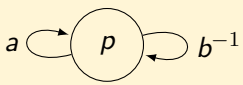

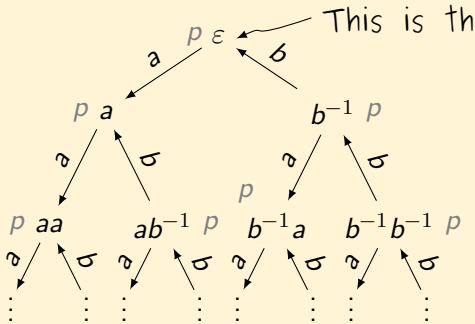

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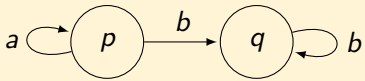

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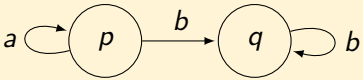

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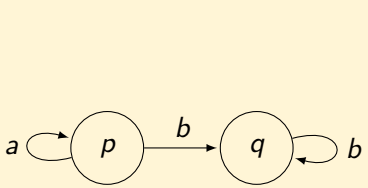

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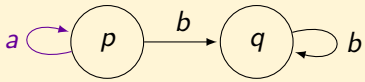

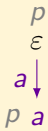

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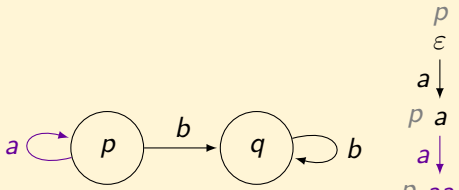

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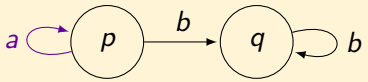

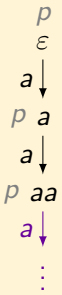

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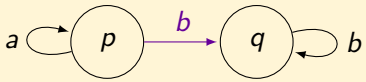

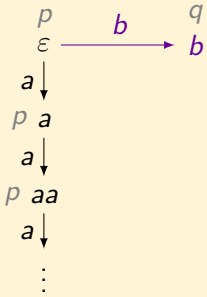

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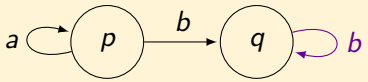

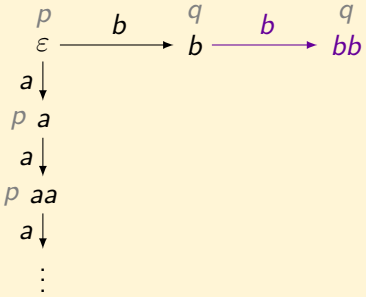

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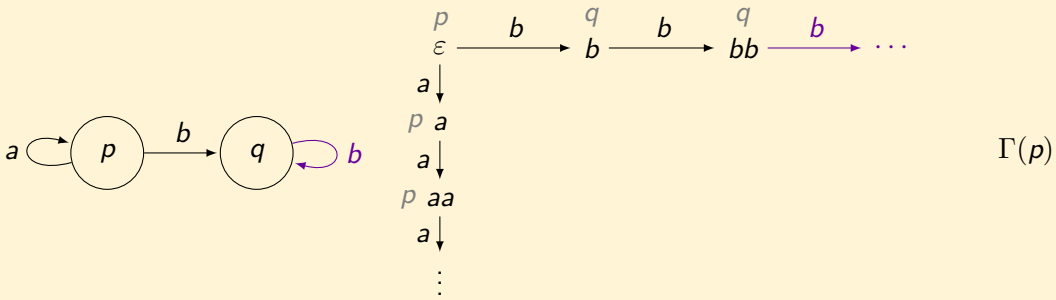

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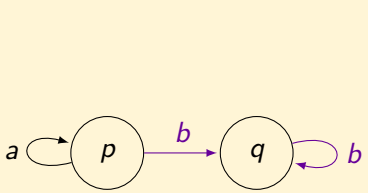

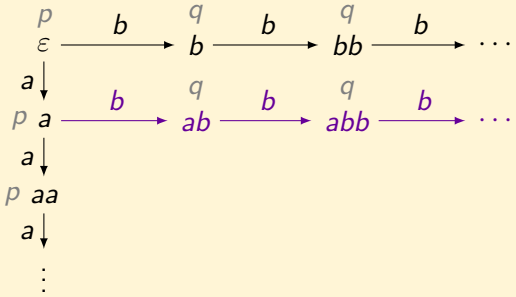

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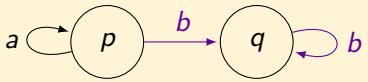

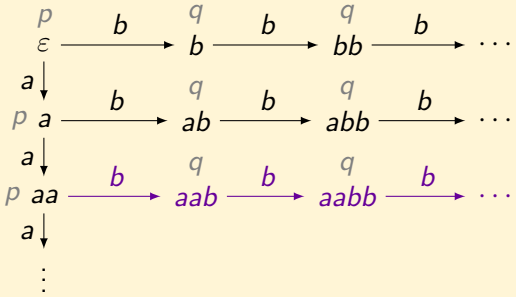

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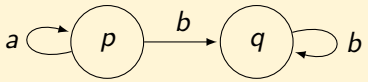

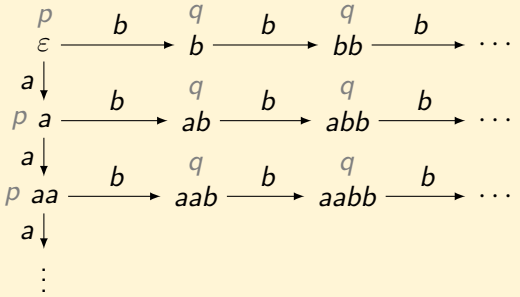

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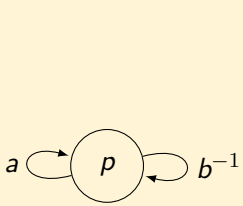

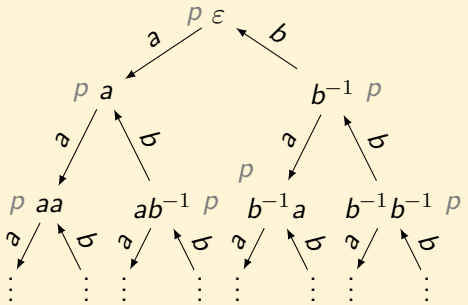

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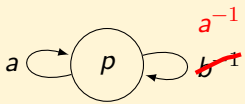

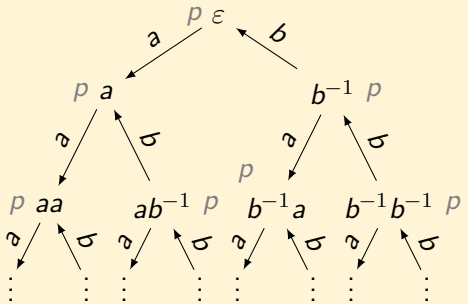

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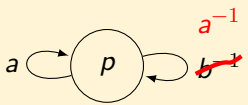

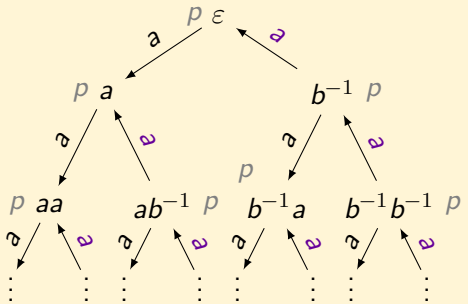

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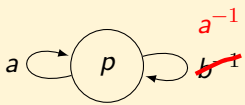

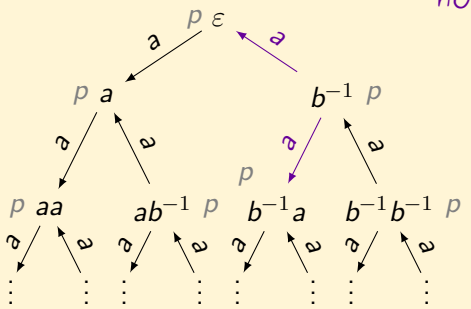

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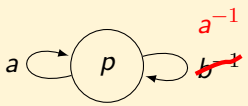

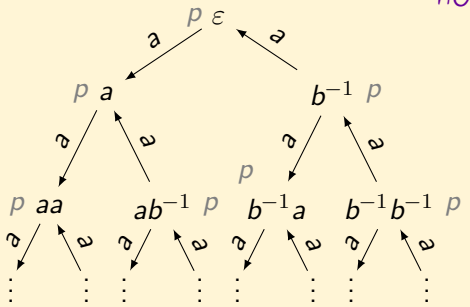

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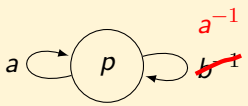

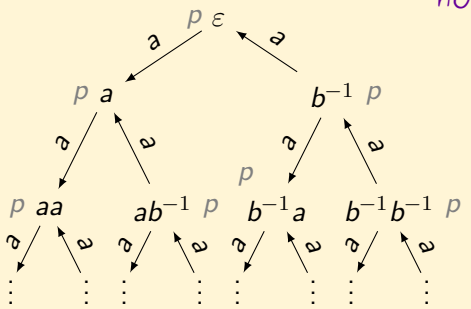

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$\iff$: the pDFA is reduced

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Γ: inverse $A^{\pm 1}$-tree, i. e. a tree as an unlabeled, undirected graph

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What can we do with this?

Schützenberger Graphs and Green's Relations

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Constant: $M = \mathrm{Inv}\langle A \mid \ell_i = r_i \rangle$ *for* $\ell_i = 1 = r_i$ *in* $F(A)$
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But: $\mathcal{D}$ and $\mathcal{J}$ remain open!

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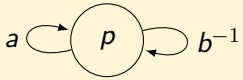

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$\mathscr{L}(p)$: words readable from state p in its pDFA

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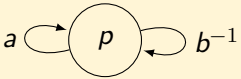

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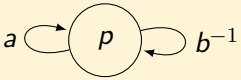

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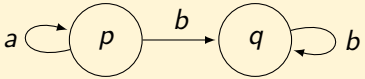

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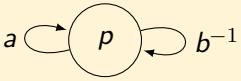

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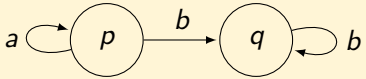

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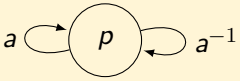

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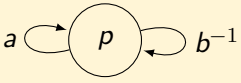

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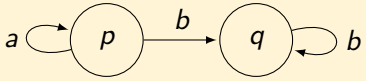

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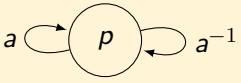

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Let p and q be states of reduced pDFAs. Then:

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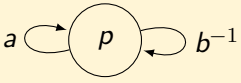

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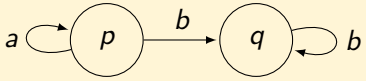

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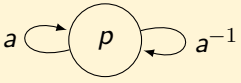

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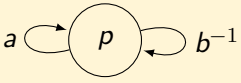

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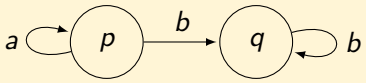

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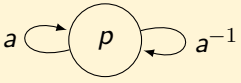

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Adapting a construction by Cho & Huynh, we may assume every state to be accessible from p and q, respectively (for the lower bound).

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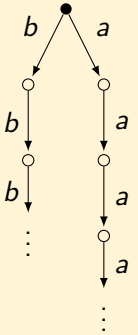

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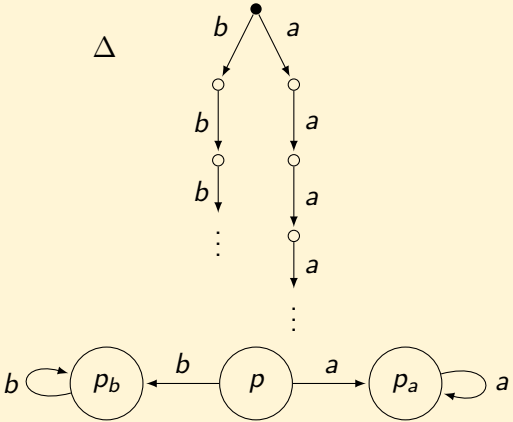

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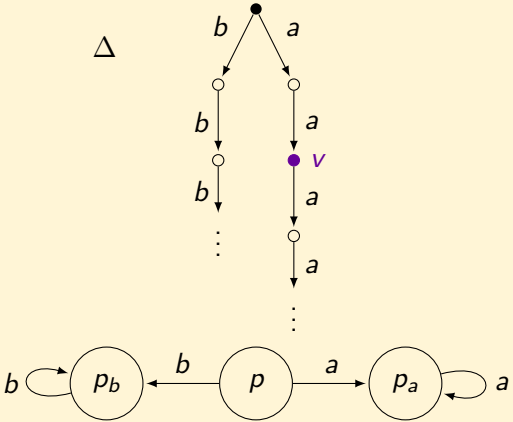

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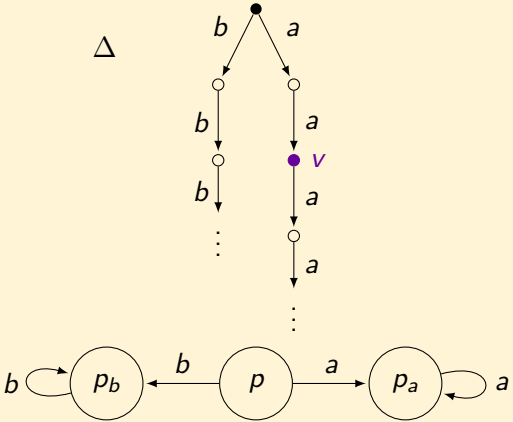

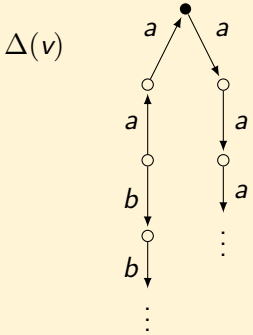

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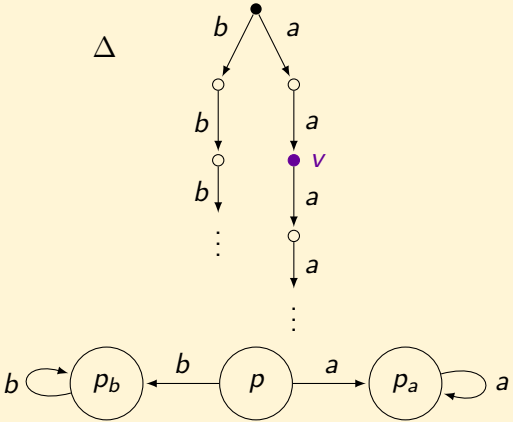
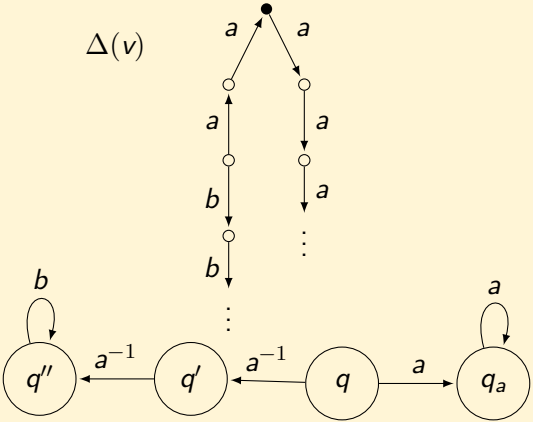
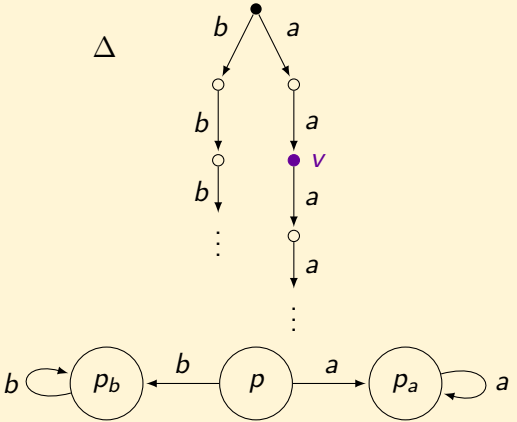
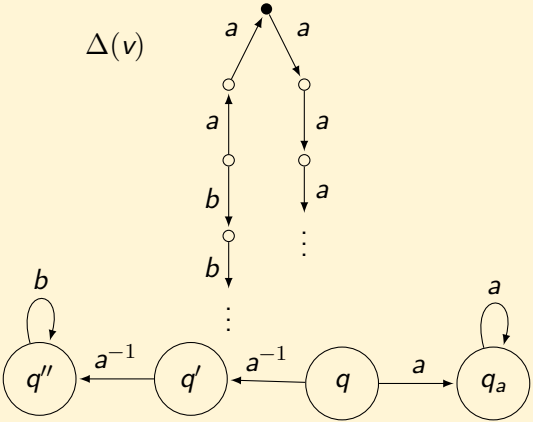




The Non-Rooted Case: The Problem

Example





The languages don't seem connected!

The Non-Rooted Isomorphism Problem

Theorem (W., arXiv 2026)

The non-rooted isomorphism problem for inverse context-free trees

Input: *states $\check{p}$ and $\check{q}$ of reduced pDFAs*
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The tricky part is "$\in \mathsf{NL} \subseteq \mathsf{P}$".

Proof Idea

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Predicate: $U(p, q)$

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Predicate: $U(p, q) \iff \exists$ node $v \in \Delta$ labeled by q

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Thus: $U(p, q) \iff (q = \check{q}$ and $\mathscr{L}(p) = \mathscr{L}(\check{q}))$

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Thus: $U(p, q) \iff (q = \check{q}$ and $\mathscr{L}(p) = \mathscr{L}(\check{q}))$ or "v is not the root"

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$\mathcal{A}$ p ε

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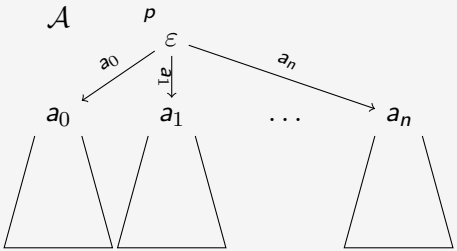

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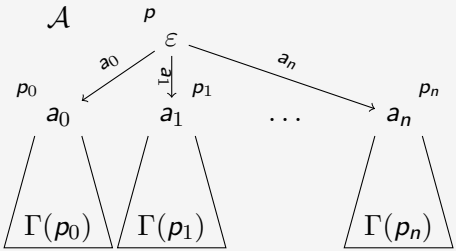

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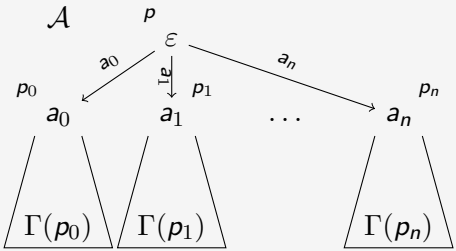

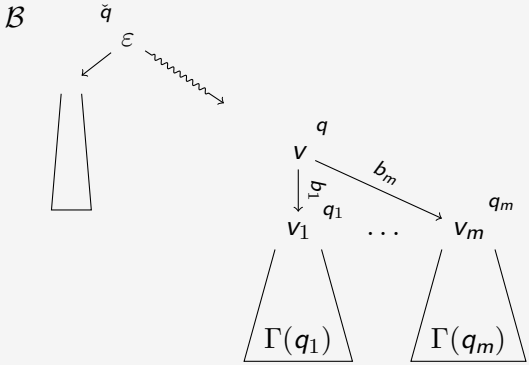

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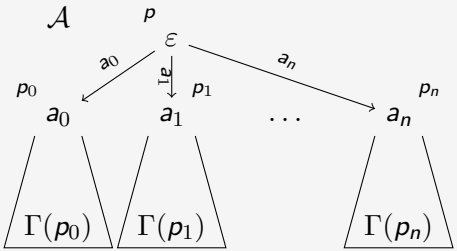

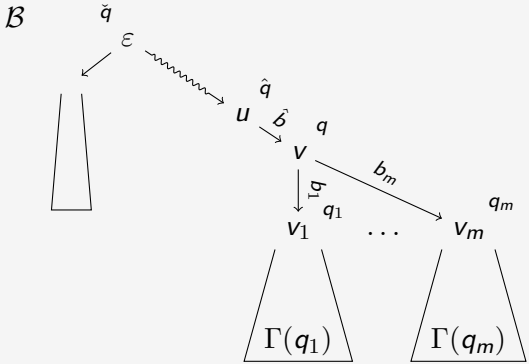

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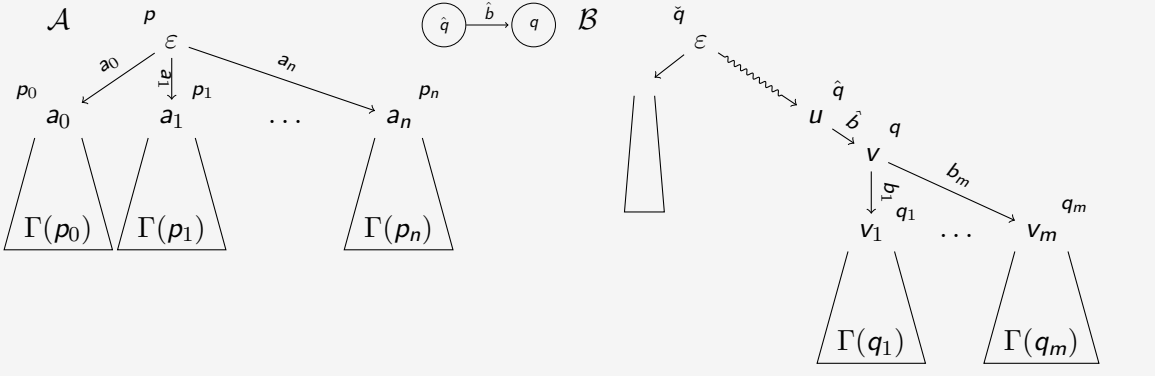

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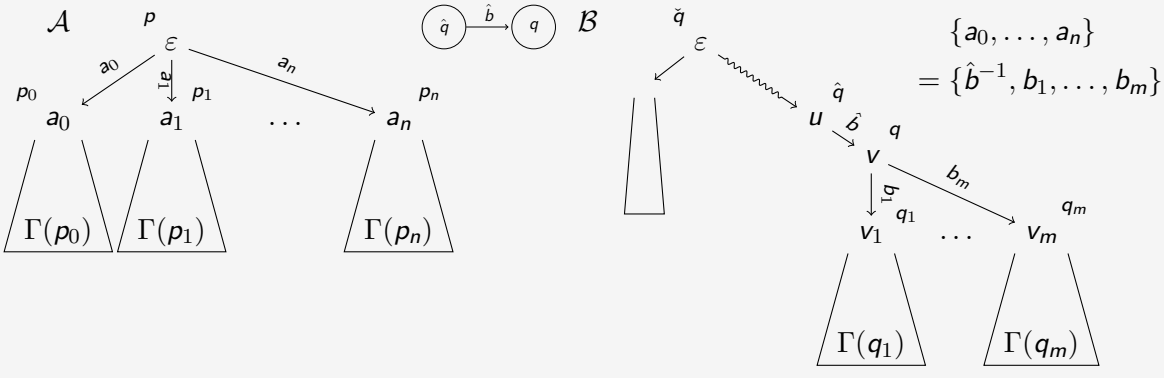

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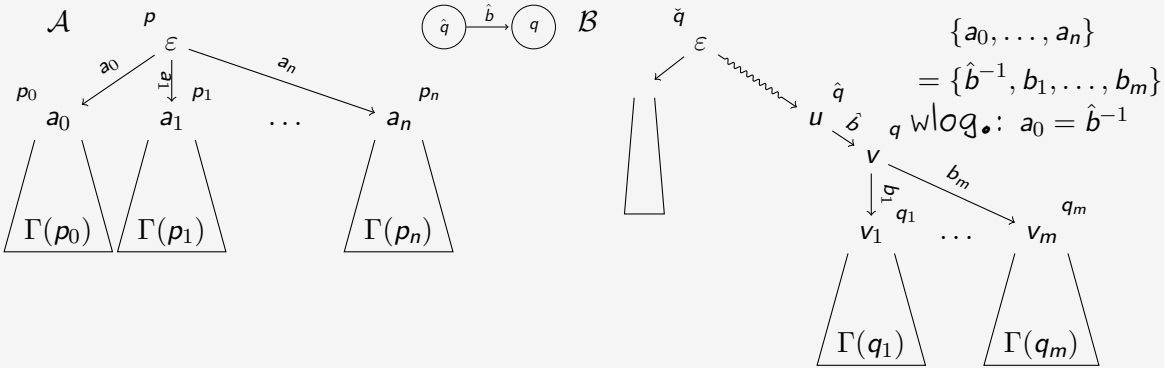

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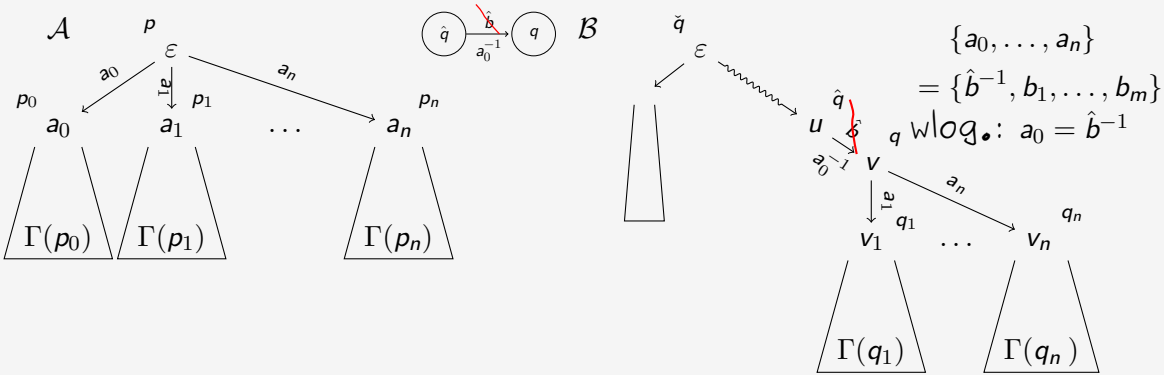

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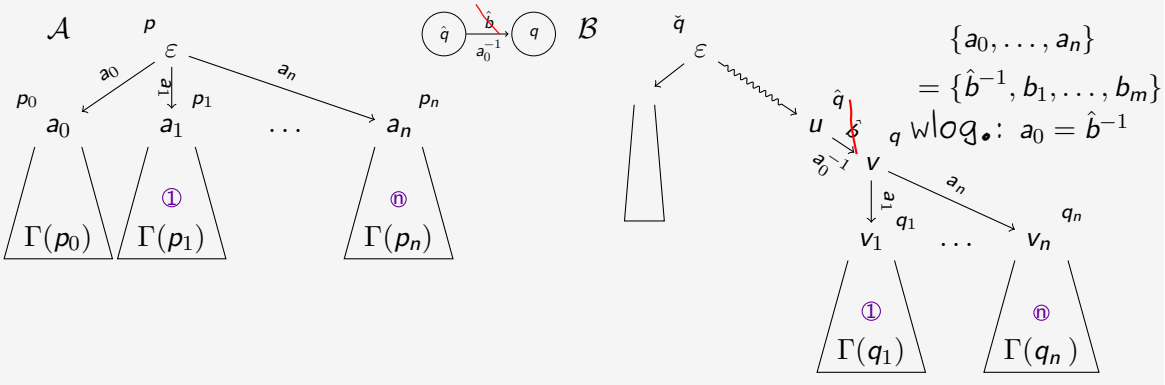

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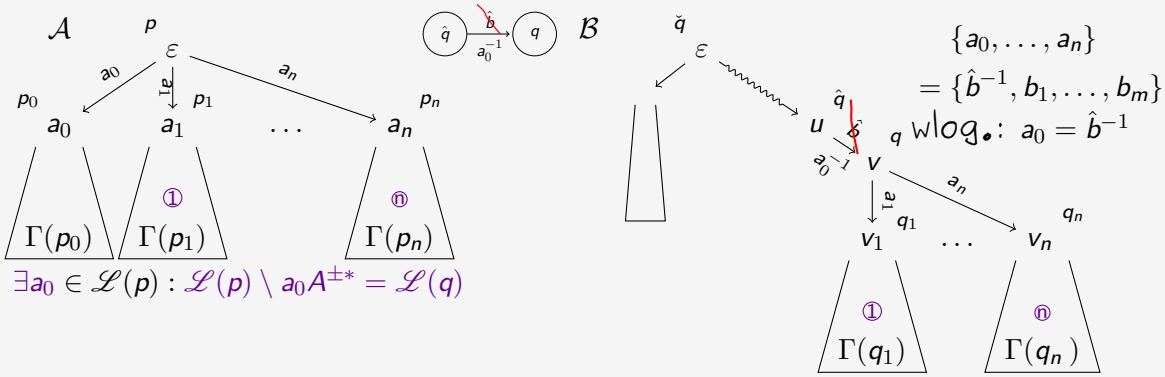

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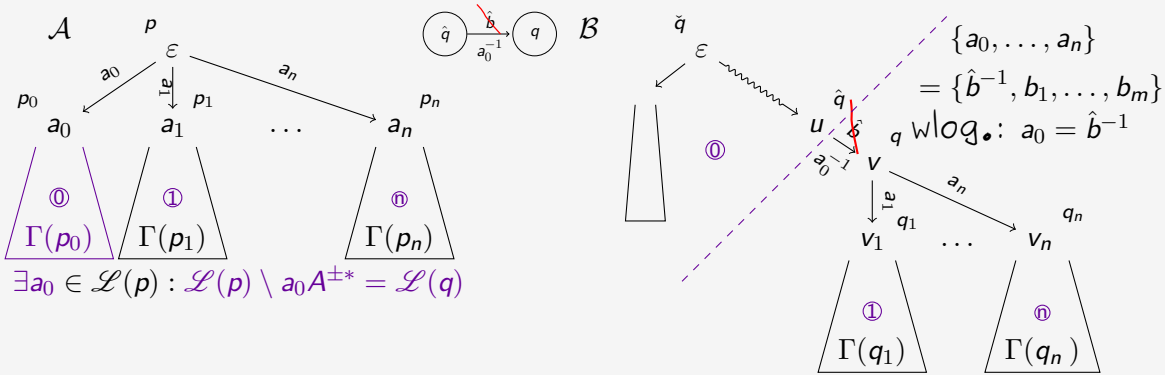

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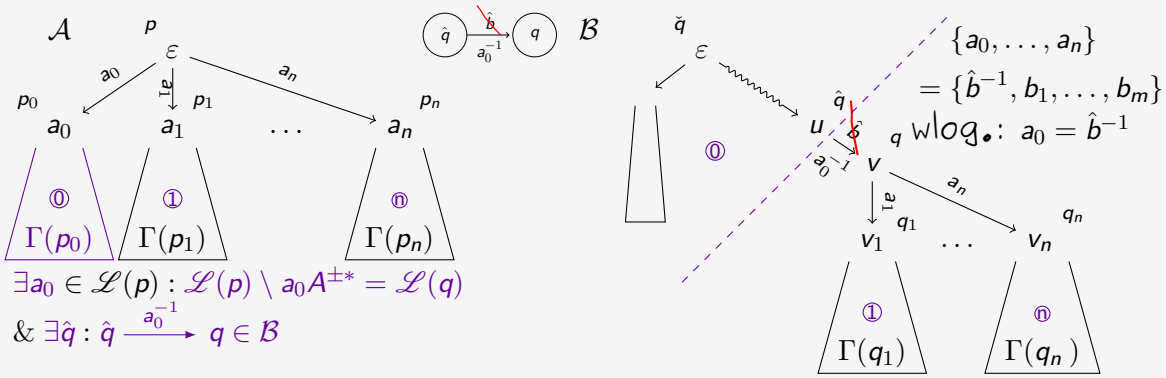

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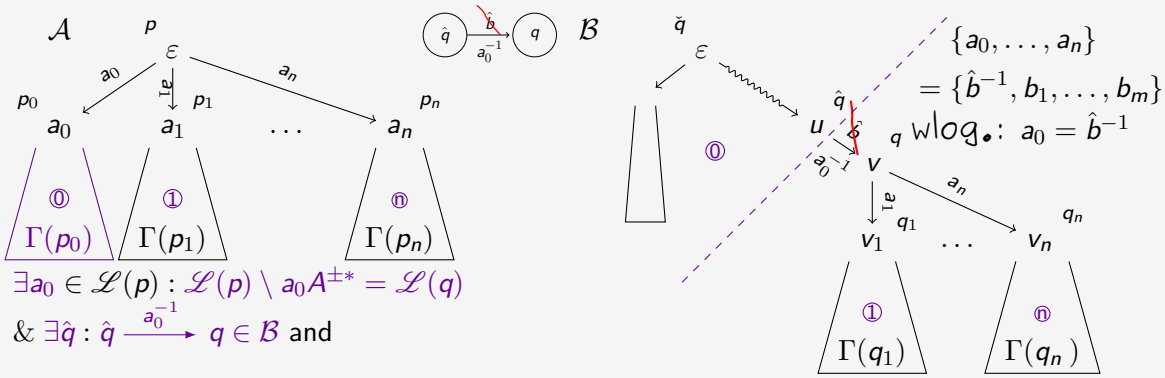

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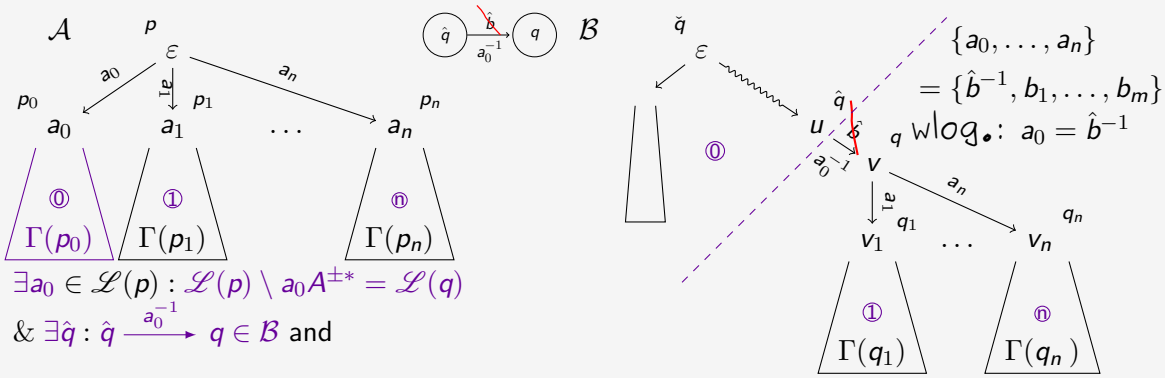

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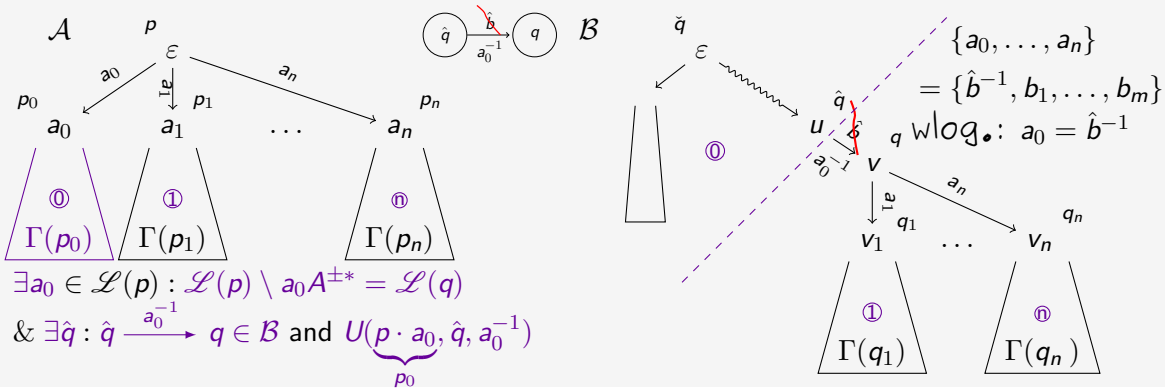

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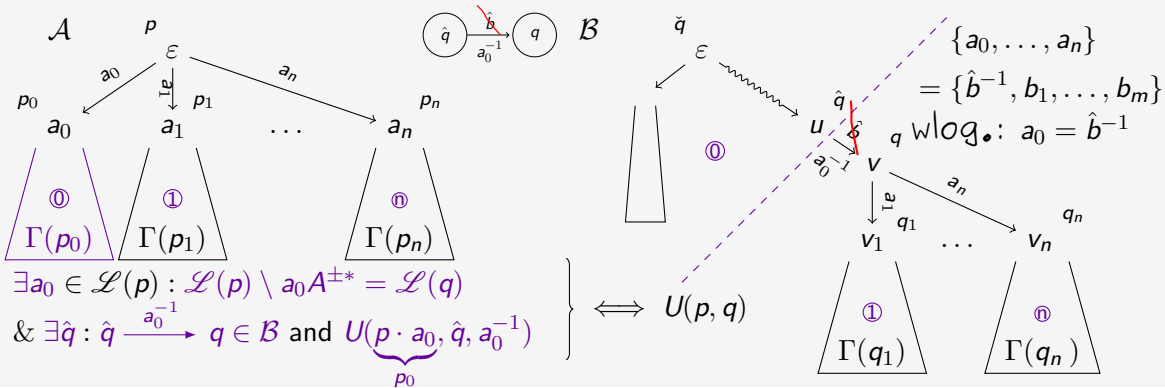

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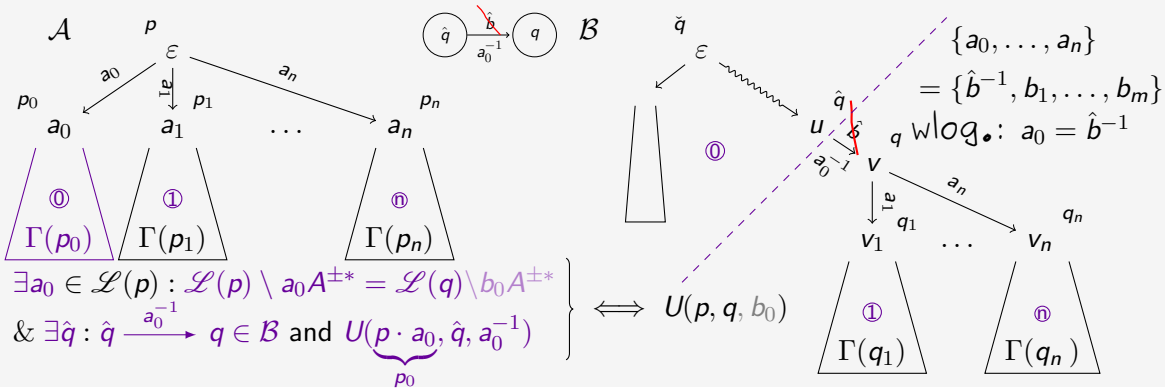

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$$U(p, q, b_0) \iff q = \check{q} \;\&\; \mathscr{L}(p) = \mathscr{L}(\check{q}) \setminus b_0 A^{\pm *}$$

or

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Summary

p: state of reduced pDFA $\mathcal{A}$, $\check{q}$: state of a reduced pDFA $\mathcal{B}$, $\Delta = \Gamma(\check{q})$

Predicate: $U(p, q, b_0) \iff \exists$ node $v \in \Delta$ labeled by $q : \Gamma(p) \doteq \Delta(v) \setminus v\, b_0 A^{\pm *}$

$$U(p, q, b_0) \iff q = \check{q} \;\&\; \mathscr{L}(p) = \mathscr{L}(\check{q}) \setminus b_0 A^{\pm *}$$
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- Check whether some $U(\check{p}, q)$ can be reached from a marked node.

Thank you!