The University of Manchester

# An Overview on Algorithmic Results for Automaton (Semi)Groups

Jan Philipp **Wächter**

Department of Mathematics
University of Manchester

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23 March 2026

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How can we encode an infinite (semi)group?

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- traditional presentation: $\langle q_1, \dots, q_n \mid \ell_1 = r_1, \dots, \ell_m = r_m \rangle$
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- We will use: automata

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Let's look at a famous example!

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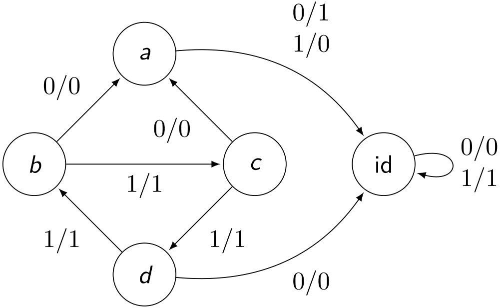

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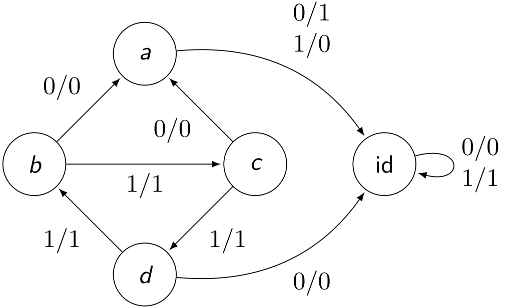


"How does this work?"

⇝ next slides

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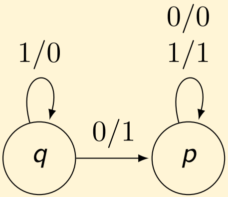

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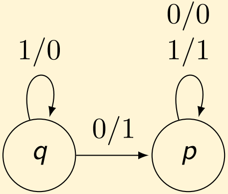

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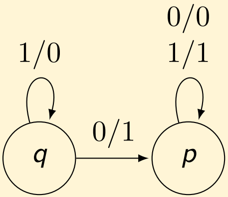

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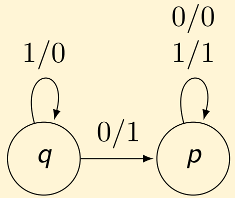

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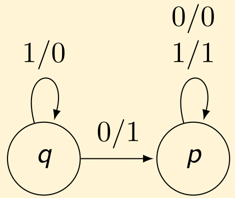

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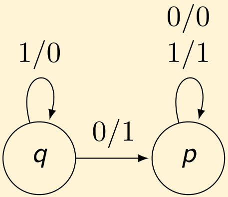

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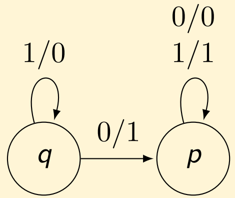

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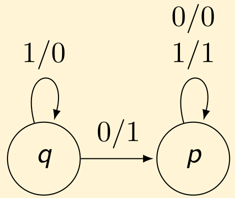

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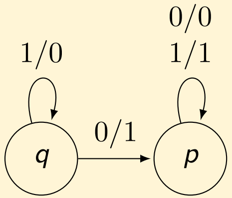

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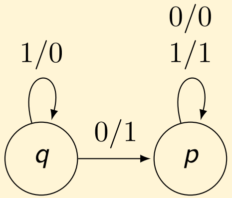

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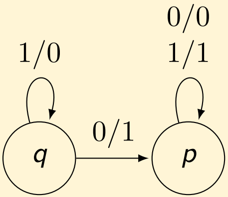

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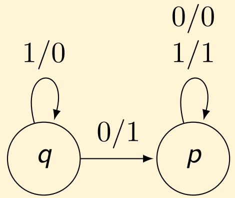


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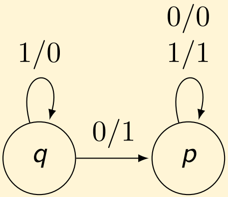


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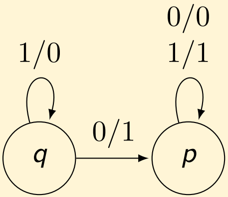


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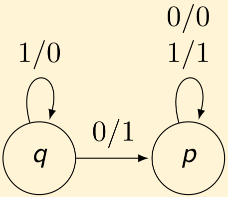


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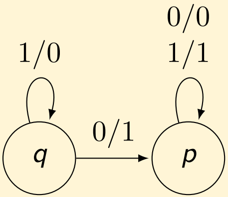


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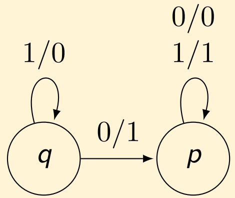


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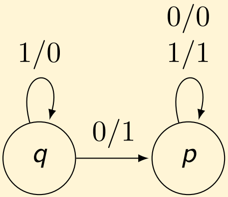


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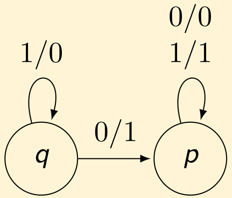


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- for $\mathscr{G}$-automata: bijections

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- monoid $\mathscr{M}(\mathcal{T})$ generated by $\mathcal{T}$: closure under composition of the functions induced by the states plus identity "automaton monoid"

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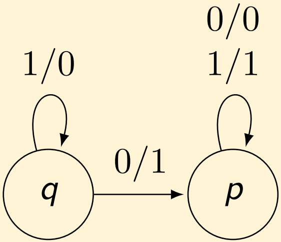


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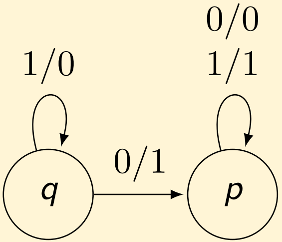


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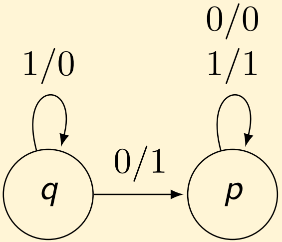


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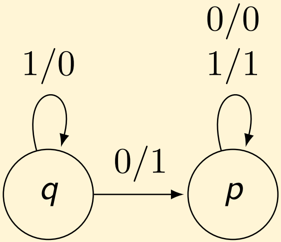


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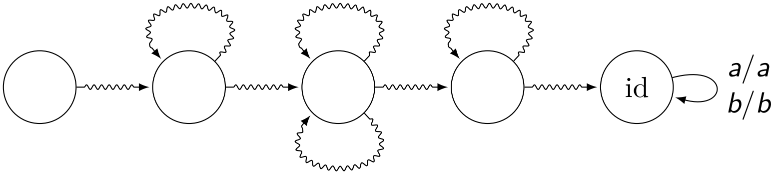

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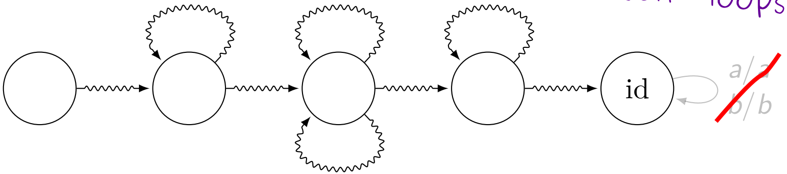

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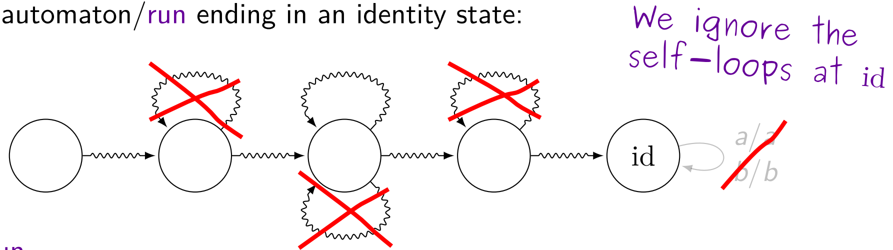


If every such run...

...has at most one cycle, the automaton has bounded activity.

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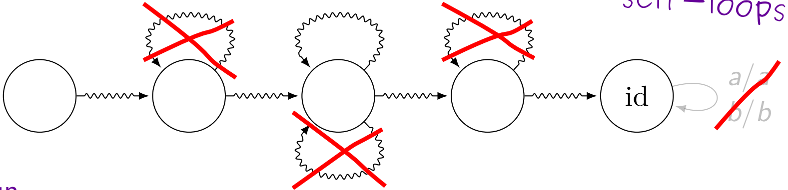


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# Sidki's Activity for Automata

Consider a subautomaton/run ending in an identity state:

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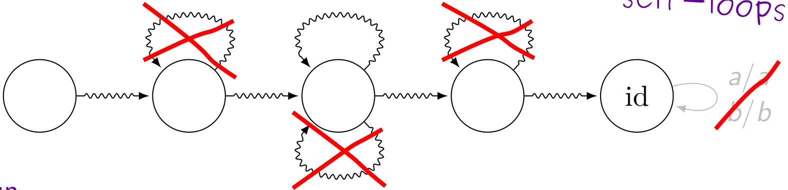


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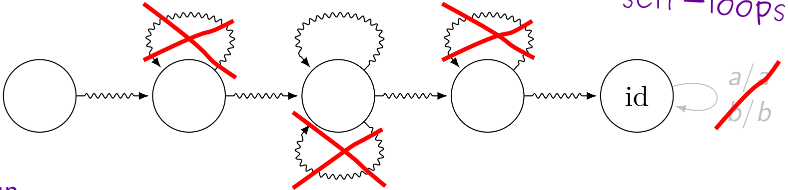


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There is a generalization to monoids (Bartholdi, Godin, Klimann, Picantin; 2018)

but without the above geometric characterization!

# Word Problem

| | general automaton groups | bounded automaton groups | general complete automaton monoids | bounded automaton monoids |
|---|---|---|---|---|
| word problem | | | | |

## Definition (Word Problem)

The word problem of an automaton groups is the problem

**Constant:** a $\mathscr{G}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$
**Input:** $\boldsymbol{p} \in Q^{\pm *}$
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| isomorphism problem | undecidable<br>follows from ŠV | *open* | undecidable<br>by group case | *open* |

# Further Problems

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|---|---|---|---|---|

# Further Problems

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## Definition (Order Problem)

The order problem of an automaton group is:

**Constant:** a $\mathscr{G}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$
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# Further Problems

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Bondarenko$^2$, Sidki, Zapata

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Bartholdi, Godin, Klimann, Picantin

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| finiteness problem | | | | |

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The finiteness problem of an automaton group/monoid is the problem

**Input:** a $\mathscr{G}/\mathscr{S}$-automaton $\mathcal{T}$

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## Theorem (Sidki; 2004)

*An automaton group of polynomial activity cannot contain a free subgroup of rank* $2$.

# Further Problems

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**Theorem (Sidki; 2004)**

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Thank you!

# Summary

| | general automaton groups | bounded automaton groups | finitary automaton groups | general complete automaton monoids | bounded automaton monoids |
|---|---|---|---|---|---|
| word problem | PSPACE-complete<br>W., Weiß; 2020 | ∈ LOGSPACE<br>$NC^1$-hard<br>by finitary case | regular<br>$NC^1$-complete<br>Barrington; 1989 | PSPACE-complete<br>D'A RW 2017<br>by group case | *open*<br>∈ PSPACE<br>$NC^1$-hard |
| uniform word problem | PSPACE-complete<br>by non-unif. case | *open*<br>∈ PSPACE<br>coNP-hard | coNP-complete<br>Kotowsky, W.; 2023 | PSPACE-complete<br>by non-unif. case | *open*<br>∈ PSPACE<br>coNP-hard |
| conjugacy problem | undecidable<br>Šunić, Ventura; '12 | *open* | $NC^1$-complete<br>by word problem | *n/a* | *n/a* |
| isomorphism problem | undecidable<br>follows from ŠV | *open* | *trivial*<br>decidable<br>complexity *open* | undecidable<br>by group case | *open* |
| order/torsion problem | undecidable<br>Gillibert; 2018<br>Barth., Mitro.; 2020 | decidable<br>$B^2SZ$; 2013 | $NC^1$-complete<br>by word problem | undecidable<br>Gillibert; 2014 | decidable<br>BGKP; 2018 |
| finiteness problem | *open* | decidable<br>Bodarenko, W.; 2021 | *trivial*<br>decidable | undecidable<br>Gillibert; 2014 | decidable<br>D'A RW; arXiv 2024 |
| freeness problem | *open* | decidable<br>by order & finit. prob. | *trivial*<br>decidable | undecidable<br>D'A RW; 2024 | *open* |