An Overview on Algorithmic Results for Automaton (Semi)Groups

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This visit is funded by EPSRC project EP/Y008626/1.

23 March 2026

Algorithmic Problems in Group Theory

Max Dehn's three fundamental problems of algorithmic group theory:

Definition (Word Problem)

Constant: a group G
Input: a group element $g \in G$
Question: is $g = \mathbb{1}$?

Definition (Conjugacy Problem)

Constant: a group G
Input: two group elements $g, h \in G$
Question: $\exists k \in G : g = k^{-1}hk$ (i. e. are they conjugate)?

Definition (Isomorphism Problem)

Input: two groups G and H
Question: are G and H isomorphic?

How can we encode an infinite (semi)group?

Encoding (Semi)Groups

- traditional presentation: $\langle q_1, \ldots, q_n \mid \ell_1 = r_1, \ldots, \ell_m = r_m \rangle$
  $Q = \{q_1, \ldots, q_n\}$: generators, $(\ell_1, r_1), \ldots (\ell_m, r_m) \in Q^+ \times Q^+$: relations
  $\rightsquigarrow$ if both sets are finite: finitely presented (semi)group
  Here: Any "reasonable" property is undecidable (Markov properties/Adian-Rabin theorem)
- Alternatives: Cayley table, permutations, invertiable matrices, ...
- We will use: automata

Why use automata?

- Many examples of (in particular) groups with interesting properties arise in this way and
- it allows for a finite description of possibly non-finitely presented (semi)groups.

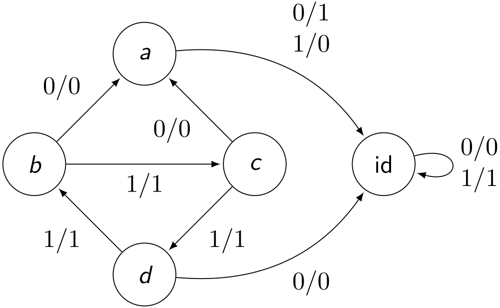
  ⇝ This makes them algorithmically interesting!

Let's look at a famous example!

Example: Grigorchuk's Group…

- …is the historically first example of a group of intermediate growth (i. e. subexponential but superpolynomial).
  "Milnor Problem"
- …is a Burnside group.
  "Burnside Problem"
- …is amenable but not elementary amenable. "Day Problem"
- …is not finitely presented.

- But: It is generated by the automaton:



"How does this work?"
$\rightsquigarrow$ next slides

Automata

Here: An automaton $\mathcal{T} = (Q, \Sigma, \delta)$ is a

- finite-state,
- letter-to-letter

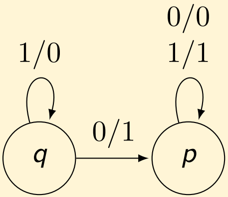
transducer

- without final or initial states.

An automaton may be:

	$\mathscr{S}$-automaton "semigroup"	$\mathscr{G}$-automaton "group"
deterministic	✓	✓
complete	(✓)	✓
invertible	–	✓

Example



"Adding Machine"

Every state q induces a function $\Sigma^* \to \Sigma^*$

$u \mapsto q \circ u$

- p induces the identity.
- $q \circ 000 = 100$

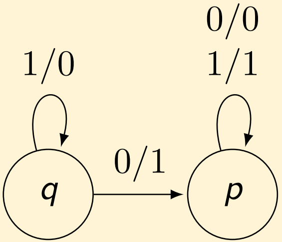
  $qq \circ 000 = q \circ 100 = 010$

  $qqq \circ 000 = \cdots = 110 \rightsquigarrow$ binary increment
- for $\mathscr{G}$-automata: bijections

Automaton Monoid and Groups

- monoid $\mathscr{M}(\mathcal{T})$ generated by $\mathcal{T}$: closure under composition of the functions induced by the states plus identity "automaton monoid"
- group $\mathscr{G}(\mathcal{T})$ generated by $\mathcal{T}$: include inverse functions "automaton group"

Example



- p: identity
- q: increment

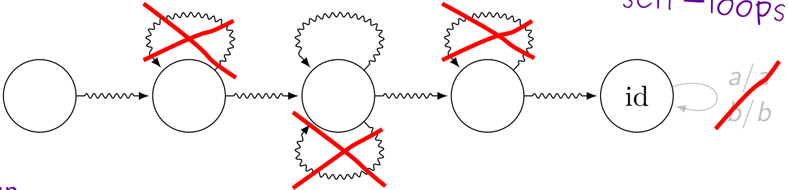
$\mathscr{M}(\mathcal{T}) \simeq \mathbb{N}_0$

$\mathscr{G}(\mathcal{T}) \simeq \mathbb{Z}$

Sidki's Activity for Automata

Consider a subautomaton/run ending in an identity state:

We ignore the self-loops at id



If every such run...

...has at most one cycle, the automaton has bounded activity. e.g. Grigorchuk's group

There is a generalization to monoids (Bartholdi, Godin, Klimann, Picantin; 2018)

but without the above geometric characterization!

Word Problem

	general automaton groups	bounded automaton groups	general complete automaton monoids	bounded automaton monoids
word problem	PSpace-complete W., Weiß; 2020	∈ LogSpace NC^1-hard follows f. Barrington; 1989	PSpace-complete D'A RW 2017 by group case	*open* ∈ PSpace NC^1-hard

Definition (Word Problem)

The word problem of an automaton groups is the problem

Constant: a $\mathscr{G}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$
Input: $\boldsymbol{p} \in Q^{\pm *}$
Question: is $\boldsymbol{p} = \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$?

For monoids:
Constant: $\mathscr{S}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$
Input: $\boldsymbol{p}, \boldsymbol{q} \in Q^*$
Question: $\boldsymbol{p} = \boldsymbol{q}$ in $\mathscr{M}(\mathcal{T})$?

Dehn's Problems

	general automaton groups	bounded automaton groups	general complete automaton monoids	bounded automaton monoids
word problem	PSPACE-complete W., Weiß; 2020	$\in$ LOGSPACE NC^1-hard follows f. Barrington; 1989	PSPACE-complete D'A RW 2017 by group case	*open* $\in$ PSPACE NC^1-hard
conjugacy problem	undecidable Šunić, Ventura; 2012	*open*	*n/a*	*n/a*
isomorphism problem	undecidable follows from ŠV	*open*	undecidable by group case	*open*

Further Problems

	general automaton groups	bounded automaton groups	general complete automaton monoids	bounded automaton monoids
order/torsion problem	undecidable Gillibert; 2018 Barth., Mitro.; 2020	decidable B²SZ; 2013	undecidable Gillibert; 2014	decidable BGKP; 2018

Definition (Order Problem)

The order problem of an automaton group is:

Constant: a $\mathscr{G}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$

Input: $\boldsymbol{p} \in Q^{\pm *}$

Question: $\exists n > 0 : \boldsymbol{p}^n = \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$?

Definition (Torsion Problem)

The torsion problem of an aut. monoid is:

Constant: a $\mathscr{S}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$

Input: $\boldsymbol{p} \in Q^*$

Question: $\exists m, n : \boldsymbol{p}^{m+n} = \boldsymbol{p}^m$ in $\mathscr{M}(\mathcal{T})$?

Further Problems

	general automaton groups	bounded automaton groups	general complete automaton monoids	bounded automaton monoids
order/torsion problem	undecidable Gillibert; 2018 Barth., Mitro.; 2020	decidable B²SZ; 2013	undecidable Gillibert; 2014	decidable BGKP; 2018
finiteness problem	*open*	decidable Bodarenko, W.; 2021	undecidable Gillibert; 2014	decidable D'A RW; arXiv 2024

Definition (Finiteness Problem)

The finiteness problem of an automaton group/monoid is the problem

Input: a $\mathscr{G}/\mathscr{S}$-automaton $\mathcal{T}$

Question: is $\mathscr{G}(\mathcal{T})/\mathscr{M}(\mathcal{T})$ finite?

Further Problems

	general automaton groups	bounded automaton groups	general complete automaton monoids	bounded automaton monoids
order/torsion problem	undecidable Gillibert; 2018 Barth., Mitro.; 2020	decidable B^2SZ; 2013	undecidable Gillibert; 2014	decidable BGKP; 2018
finiteness problem	*open*	decidable Bodarenko, W.; 2021	undecidable Gillibert; 2014	decidable D'A RW; arXiv 2024
freeness problem	*open*	decidable almost trivial	undecidable D'A RW; 2024	*open*

Definition (Freeness Problem)

The freeness problem of an automaton group/monoid is the problem

Input: a $\mathscr{G}/\mathscr{S}$-automaton $\mathcal{T}$
Question: is $\mathscr{G}(\mathcal{T})/\mathscr{M}(\mathcal{T})$ free?

Thank you!

Summary

	general automaton groups	bounded automaton groups	finitary automaton groups	general complete automaton monoids	bounded automaton monoids
word problem	PSpace-complete W., Weiß; 2020	∈ LogSpace NC^1-hard by finitary case	regular NC^1-complete Barrington; 1989	PSpace-complete D'A RW 2017 by group case	*open* ∈ PSpace NC^1-hard
uniform word problem	PSpace-complete by non-unif. case	*open* ∈ PSpace coNP-hard	coNP-complete Kotowsky, W.; 2023	PSpace-complete by non-unif. case	*open* ∈ PSpace coNP-hard
conjugacy problem	undecidable Šunić, Ventura; '12	*open*	NC^1-complete by word problem	*n/a*	*n/a*
isomorphism problem	undecidable follows from ŠV	*open*	*trivial* decidable complexity *open*	undecidable by group case	*open*
order/torsion problem	undecidable Gillibert; 2018 Barth., Mitro.; 2020	decidable B^2SZ; 2013	NC^1-complete by word problem	undecidable Gillibert; 2014	decidable BGKP; 2018
finiteness problem	*open*	decidable Bodarenko, W.; 2021	*trivial* decidable	undecidable Gillibert; 2014	decidable D'A RW; arXiv 2024
freeness problem	*open*	decidable by order & finit. prob.	*trivial* decidable	undecidable D'A RW; 2024	*open*