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Context-Free Graphs in Inverse Semigroup Theory

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Department of Mathematics
University of Manchester

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Schützenberger graphs have many "desirable" properties of group Cayley graphs!

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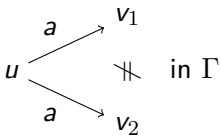

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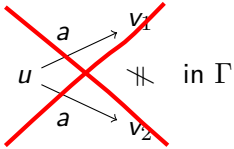

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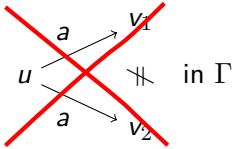

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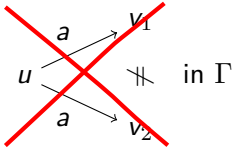

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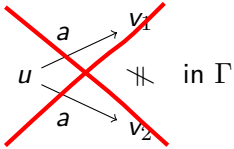

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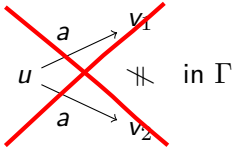

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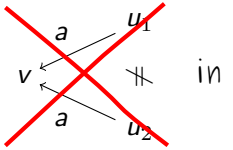

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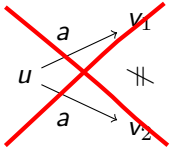

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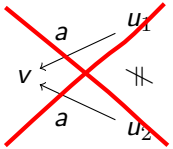

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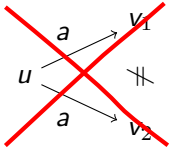

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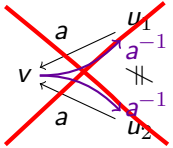

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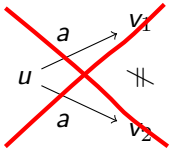

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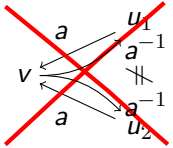

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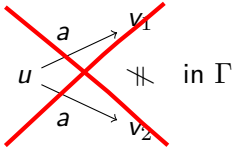

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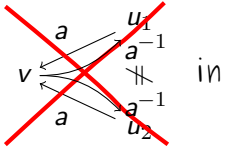

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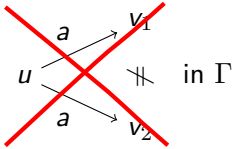

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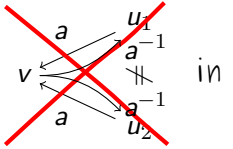

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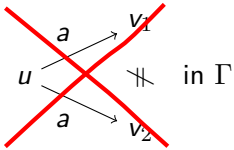

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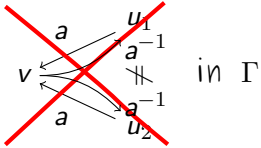

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Intermezzo!

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We could use the word problem…

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- $s \geq_{\mathcal{J}} t \iff MsM \supseteq MtM$
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If we want to decide these algorithmically, we need to

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What else can we do?

Schützenberger Graphs as Automata

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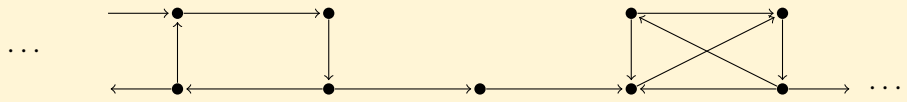

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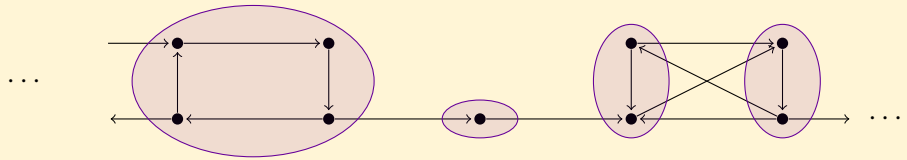

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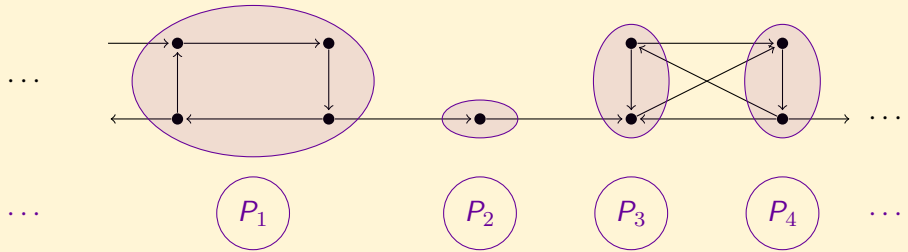

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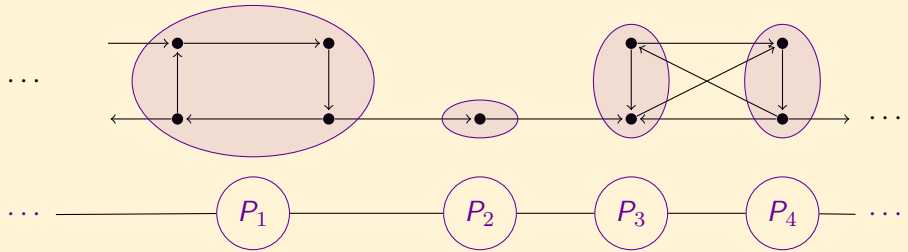

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What is a context-free language?

Context-Free Languages

For us:

$$L \subseteq A^* \text{ is context-free} \iff L = \mathscr{L}(\mathcal{C}) \text{ for some pushdown automaton } \mathcal{C}$$

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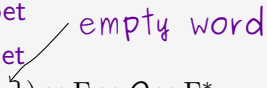

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Configuration: $w \in A^*$

remaining input word: w

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Configuration: $w \in A^*$, $p \in Q$

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state: p

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Configuration: $w \in A^*$, $p \in Q$, $\gamma' \in \Gamma^*$

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stack: γ'

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remaining input word:	$a_0 w$		w
state:	p	$\vdash_\tau$	q
stack:	$A\gamma'$		$\gamma\gamma'$

Example PDA

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states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\bot$ (bottom symbol)

transitions: $p, a, \bot \to p, A\bot$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

$q, b, A \to q, \varepsilon$ $q, \varepsilon, \bot \to f, \varepsilon$

remaining input: $aabb$

state: p

stack: $\bot$

Example PDA

Example

states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\bot$ (bottom symbol)

transitions: $p, a, \bot \to p, A\bot$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

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$q, b, A \to q, \varepsilon$ $q, \varepsilon, \bot \to f, \varepsilon$

remaining input:	$aabb$		abb
state:	p	$\vdash$	p
stack:	$\bot$		$A\bot$

Example PDA

Example

states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\bot$ (bottom symbol)

transitions: $p, a, \bot \to p, A\bot$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

$q, b, A \to q, \varepsilon$ $q, \varepsilon, \bot \to f, \varepsilon$

remaining input:	$aabb$		abb
state:	p	$\vdash$	p
stack:	$\bot$		$A\bot$

Example PDA

Example

states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\bot$ (bottom symbol)

transitions: $p, a, \bot \to p, A\bot$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

$q, b, A \to q, \varepsilon$ $q, \varepsilon, \bot \to f, \varepsilon$

remaining input:	$aabb$		abb	bb
state:	p	$\vdash$	p	$\vdash$ p
stack:	$\bot$		$A\bot$	$AA\bot$

Example PDA

Example

states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\bot$ (bottom symbol)

transitions: $p, a, \bot \to p, A\bot$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

$q, b, A \to q, \varepsilon$ $q, \varepsilon, \bot \to f, \varepsilon$

remaining input:	$aabb$		abb	bb
state:	p	$\vdash$	p $\vdash$	p
stack:	$\bot$		$A\bot$	$AA\bot$

Example PDA

Example

states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\bot$ (bottom symbol)

transitions: $p, a, \bot \to p, A\bot$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

$q, b, A \to q, \varepsilon$ $q, \varepsilon, \bot \to f, \varepsilon$

remaining input:	$aabb$		abb		bb		b
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$	q
stack:	$\bot$		$A\bot$		$AA\bot$		$A\bot$

Example PDA

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$q, b, A \to q, \varepsilon$ $q, \varepsilon, \bot \to f, \varepsilon$

remaining input:	$aabb$		abb		bb	b
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$ q
stack:	$\bot$		$A\bot$		$AA\bot$	$A\bot$

Example PDA

Example

states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\perp$ (bottom symbol)

transitions: $p, a, \perp \to p, A\perp$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

$q, b, A \to q, \varepsilon$ $q, \varepsilon, \perp \to f, \varepsilon$

remaining input:	$aabb$		abb		bb		b		ε
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$	q	$\vdash$	q
stack:	$\perp$		$A\perp$		$AA\perp$		$A\perp$		$\perp$

Example PDA

Example

states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\perp$ (bottom symbol)

transitions: $p, a, \perp \to p, A\perp$ $p, a, A \to p, AA$ $p, b, A \to q, \varepsilon$

$q, b, A \to q, \varepsilon$ $q, \varepsilon, \perp \to f, \varepsilon$

remaining input:	$aabb$		abb		bb		b		ε
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$	q	$\vdash$	q
stack:	$\perp$		$A\perp$		$AA\perp$		$A\perp$		$\perp$

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$q, b, A \to q, \varepsilon$ $q, \varepsilon, \perp \to f, \varepsilon$

remaining input:	$aabb$		abb		bb		b		ε		ε
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$	q	$\vdash$	q	$\vdash$	f
stack:	$\perp$		$A\perp$		$AA\perp$		$A\perp$		$\perp$		ε

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remaining input:	$aabb$		abb		bb		b		ε		ε
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$	q	$\vdash$	q	$\vdash$	f
stack:	$\bot$		$A\bot$		$AA\bot$		$A\bot$		$\bot$		ε

accept

Example PDA

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states: p (initial), q, f (final) **alphabet**: $\{a, b\}$ **stack symbols**: A, $\perp$ (bottom symbol)

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remaining input:	$aabb$		abb		bb		b		ε		ε
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$	q	$\vdash$	q	$\vdash$	f
stack:	$\perp$		$A\perp$		$AA\perp$		$A\perp$		$\perp$		ε

accept

$$\mathscr{L}(\mathcal{C}) = \{w \in A^* \mid \begin{pmatrix} w \\ q_0 \\ \perp \end{pmatrix} \vdash^* \begin{pmatrix} \varepsilon \\ f \\ \gamma \end{pmatrix} \text{ for } f \in F, \gamma \in \Gamma^*\}$$

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stack:	$\bot$		$A\bot$		$AA\bot$		$A\bot$		$\bot$		ε

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$$= \{a^n b^n \mid n \geq 1\}$$

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remaining input:	$aabb$		abb		bb		b		ε		ε
state:	p	$\vdash$	p	$\vdash$	p	$\vdash$	q	$\vdash$	q	$\vdash$	f
stack:	$\bot$		$A\bot$		$AA\bot$		$A\bot$		$\bot$		ε

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$\rightsquigarrow$ Configuration graph $\Gamma(\mathcal{C})$

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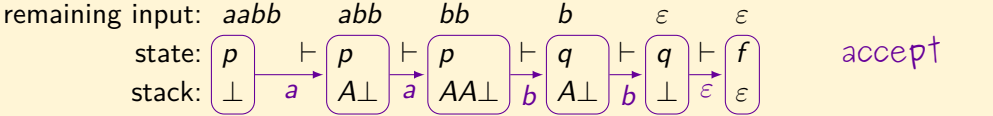

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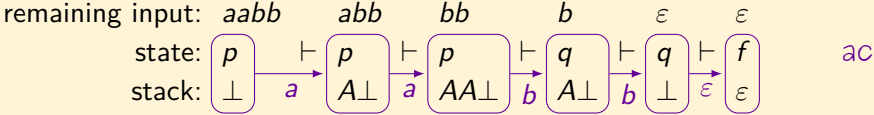

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$\rightsquigarrow$ Configuration graph $\Gamma(\mathcal{C})$ with edges $\{(p, A\gamma') \xrightarrow{a_0} (q, \gamma\gamma') \mid (p, a_0, A \to q, \gamma) \in \delta\}$

Context-Free Graphs

Γ: inverse $A^{\pm *}$-graph Then:

Γ is context-free

$\iff$ Γ is the configuration graph of some PDA (Muller, Schupp, 1990)

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Context-Free Graphs

Γ: inverse $A^{\pm *}$-graph with root u Then:

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Some details on ε-transitions and $\perp$ omitted

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Now: We may use deterministic PDAs to describe Schützenberger graphs.

But: PDAs are algorithmically not “nice”. Can we do better? Probably…

Finite Automata Generating Trees

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Definition (pDFA)

A pDFA is a finite, deterministic $A^{\pm 1}$-graph

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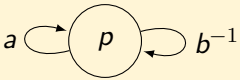
A pDFA is a finite, deterministic $A^{\pm 1}$-graph whose
nodes we call states and whose edges we call transitions.

Every state p of a pDFA generates an involutive tree via its tree of runs/path space.

Example

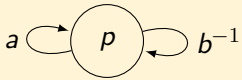
Example



Example

Example

$p\ \varepsilon$ ← This is the root!



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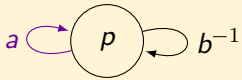

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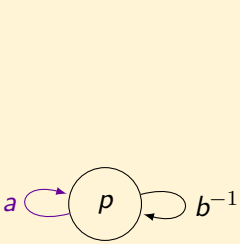

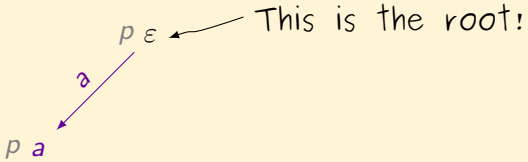

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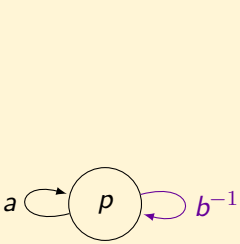

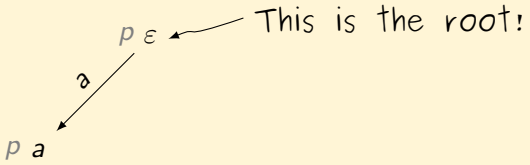

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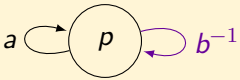

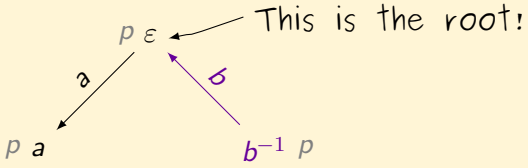

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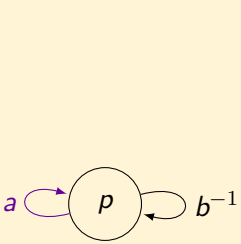

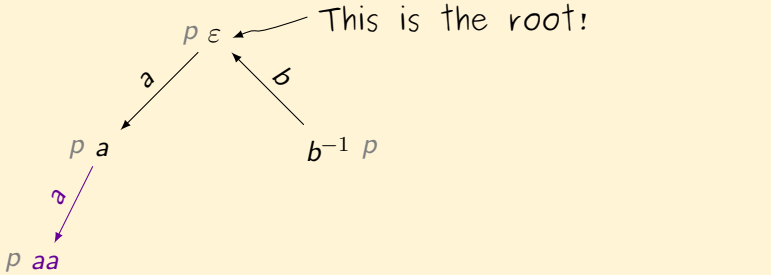

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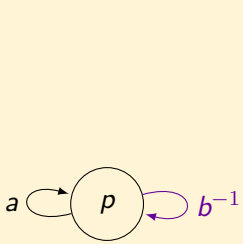

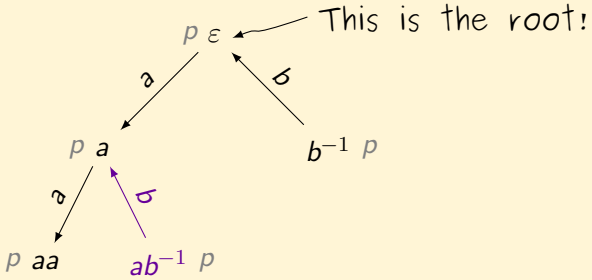

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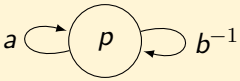

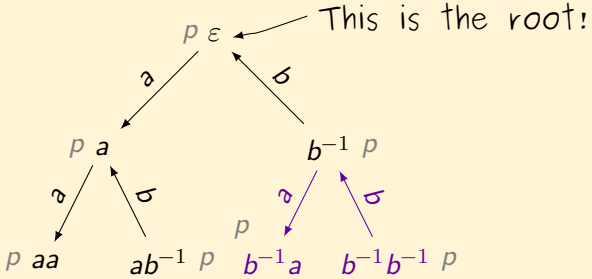

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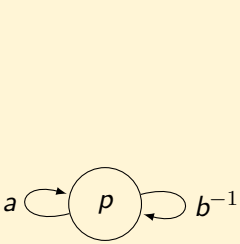

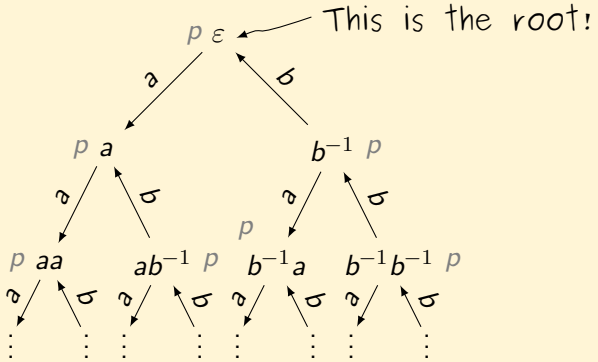

Example

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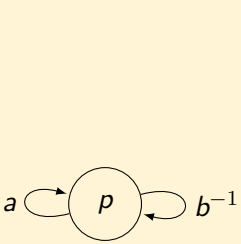

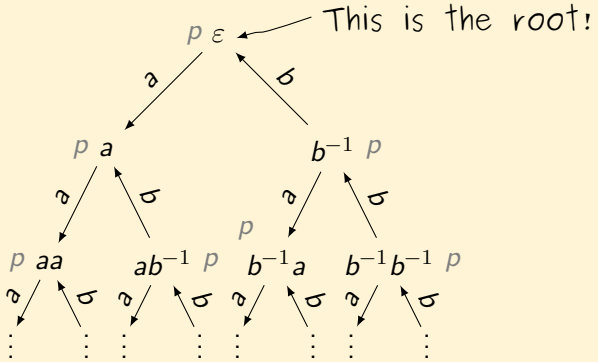

$\Gamma(p)$

Example

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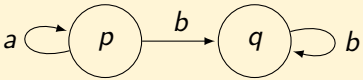

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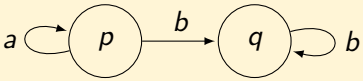

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Example

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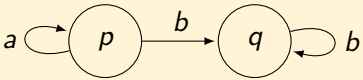

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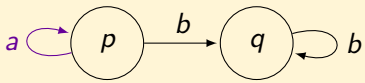

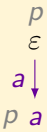

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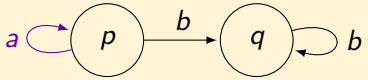

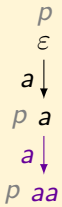

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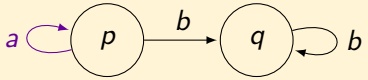

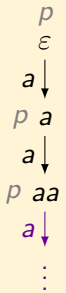

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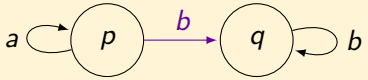

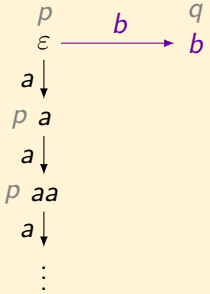

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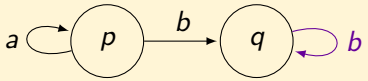

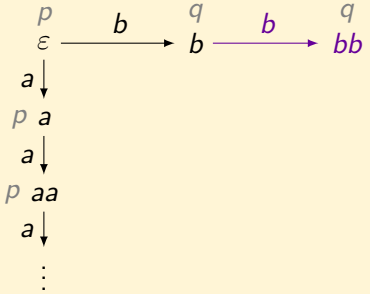

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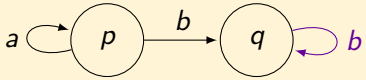

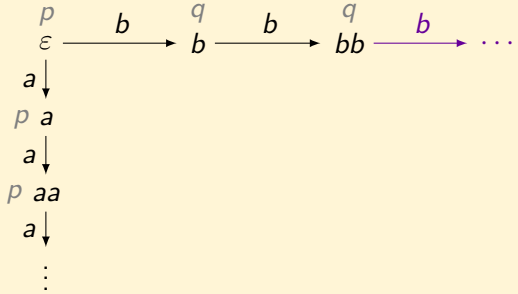

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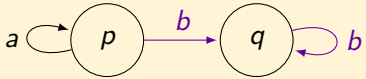

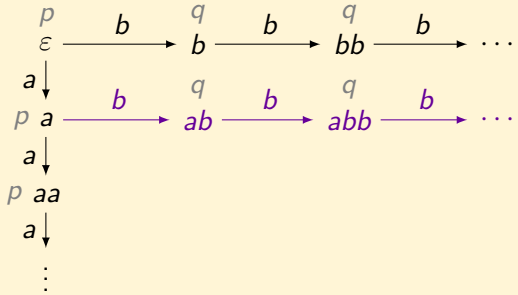

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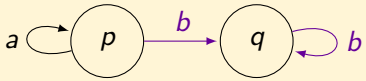

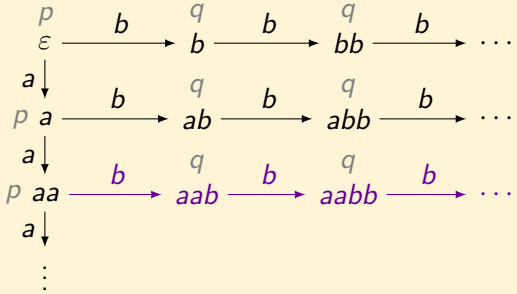

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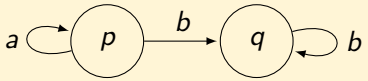

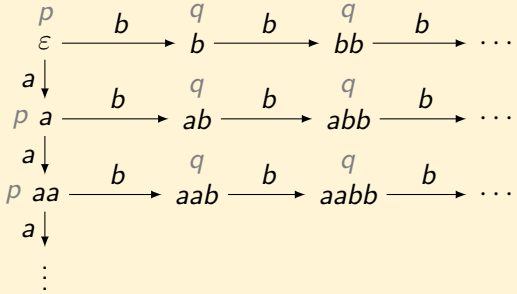

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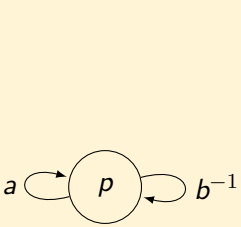

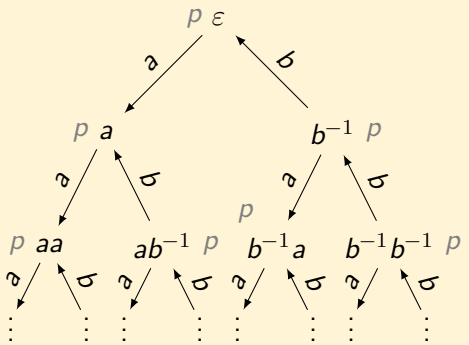

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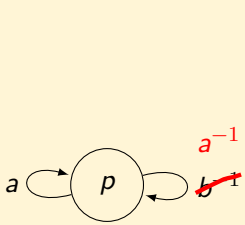

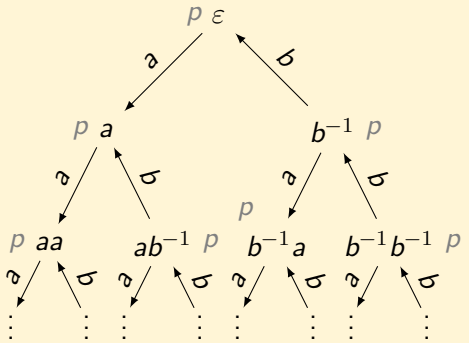

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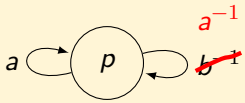

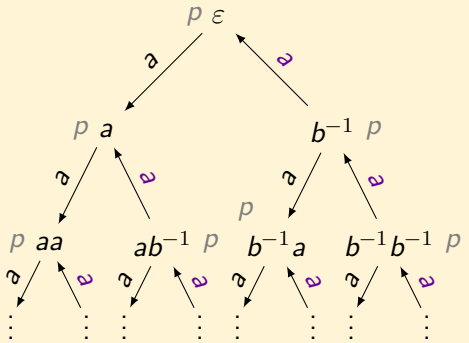

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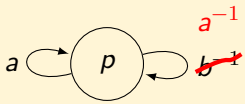

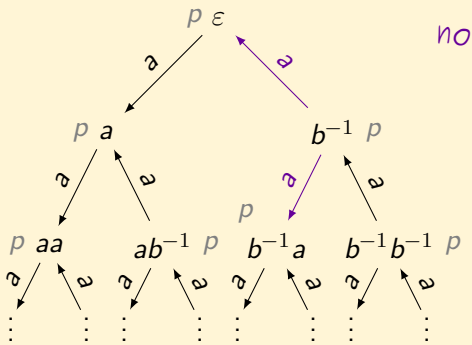

not deterministic!

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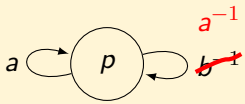

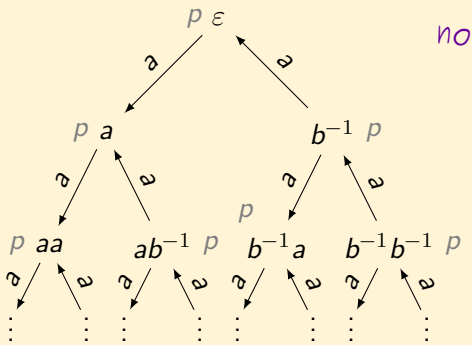

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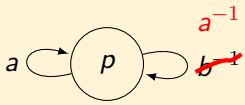

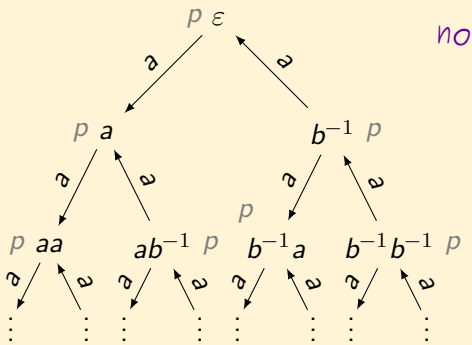

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$\iff$ the pDFA is reduced

Context-Free Trees

Γ: inverse $A^{\pm 1}$-tree, i. e. a tree as an unlabeled, undirected graph

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Context-free trees

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- hopefully serve as a stepping stone for general context-free graphs.

Inverse Monoids with Idempotent Relators

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Lemma (Margolis, Meakin, 1993)

$M = \mathrm{Inv}\langle A \mid \ell_i = r_i, i \in I \rangle$ *for* $\ell_i = 1 = r_i$ *in* $F(A)$

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The word problem

Constant: $M = \mathrm{Inv}\langle A \mid \ell_i = r_i \rangle$ *for* $\ell_i = 1 = r_i$ *in* $F(A)$
Input: $u, v \in A^{\pm *}$
Question: *is* $u = v$ *in* M*?*

is decidable.

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$\rightsquigarrow$ Let's look at the homomorphism/isomorphism problem of context−free trees!

The Rooted Isomorphism Problem

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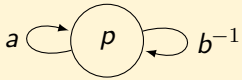

The Rooted Isomorphism Problem

$\mathscr{L}(p)$: words readable from state p in its pDFA

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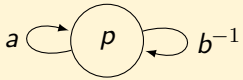

$\mathscr{L}(p) = \{a, b^{-1}\}^*$

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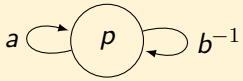

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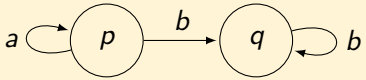

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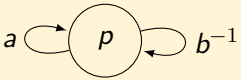

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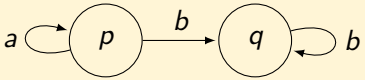

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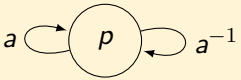

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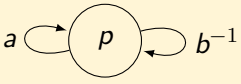
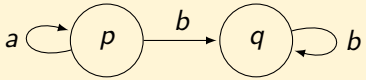
$\mathscr{L}(p)$: words readable from state p in its pDFA

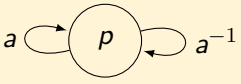
Example



$\mathscr{L}(p) = \{a, b^{-1}\}^*$



$\mathscr{L}(p) = a^* b^*$



$\mathscr{L}(p) = \{a, a^{-1}\}^*$

Fact

Let p and q be states of reduced pDFAs. Then:

$$\Gamma(p) \cong \Gamma(q) \iff \mathscr{L}(p) = \mathscr{L}(q)$$

isomorphic as **rooted** *graphs*

The Rooted Isomorphism Problem

$\mathscr{L}(p)$: words readable from state p in its pDFA

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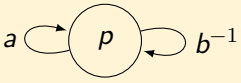

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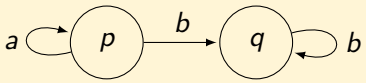

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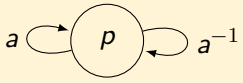

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isomorphic as **rooted** *graphs*

It is well known that the latter is in NL and, thus, polynomial time.

The Non-Rooted Case: The Problem

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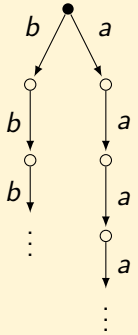
Example

Δ



The Non-Rooted Case: The Problem

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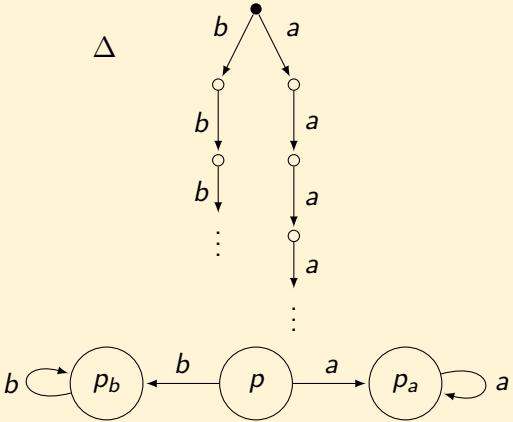

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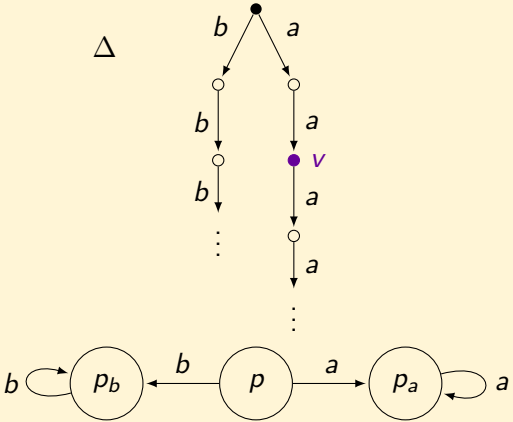

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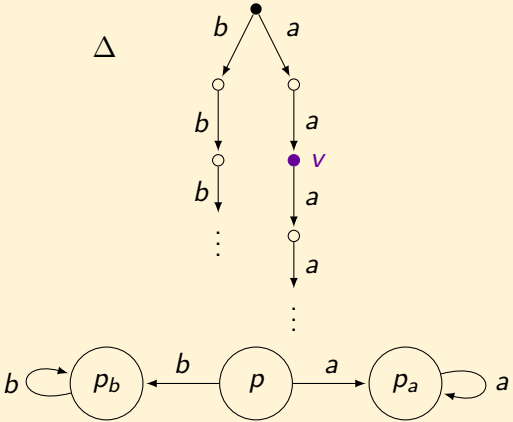

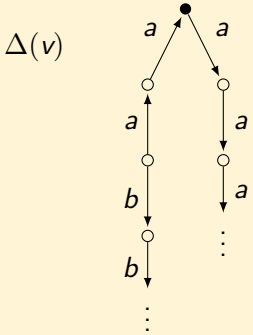

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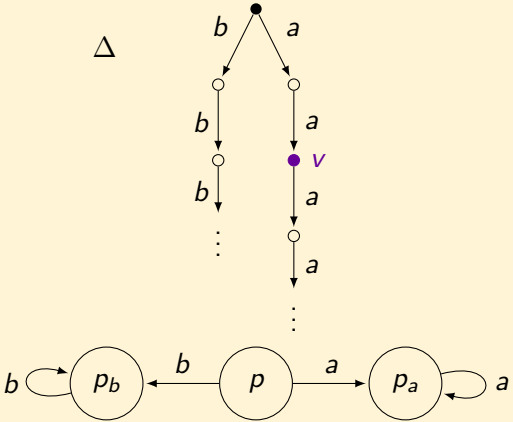

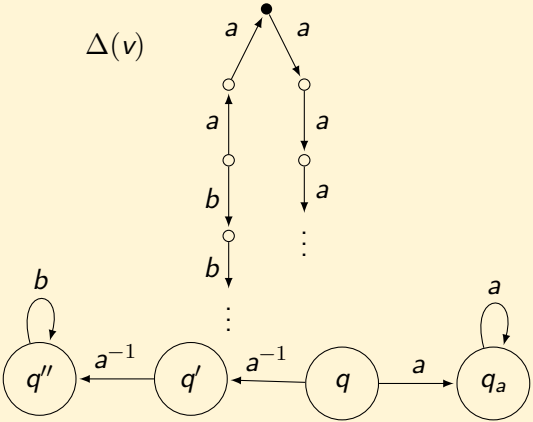

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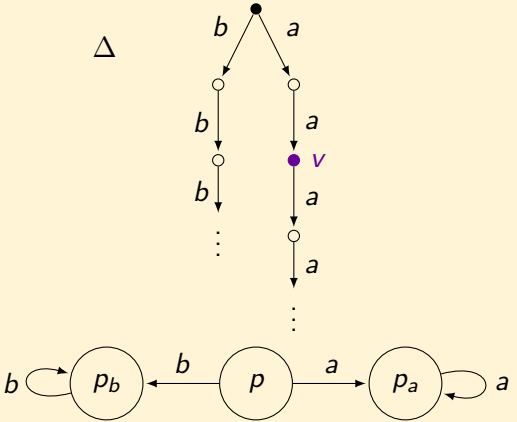

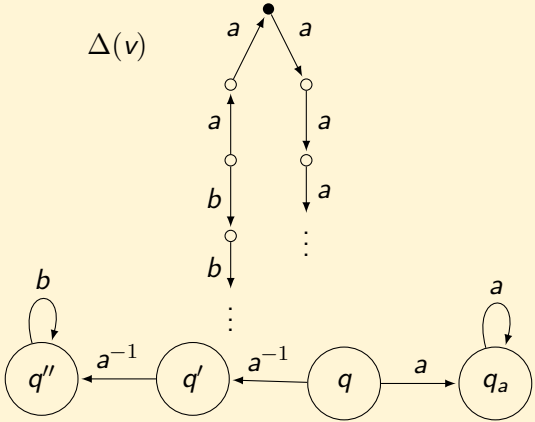

The languages don't seem connected!

The Non-Rooted Isomorphism Problem

Theorem (W., arXiv 2026)

The non-rooted isomorphism problem for inverse context-free trees

Input: *states $\check{p}$ and $\check{q}$ of reduced pDFAs*

Question: *are $\Gamma(\check{p})$ and $\Gamma(\check{q})$ isomorphic as non-rooted graphs*

is in NL and, thus, in particular, decidable in deterministic polynomial time.

Outlook

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- We still need to compute a generating reduced pDFA for $S\Gamma(s)$ from s

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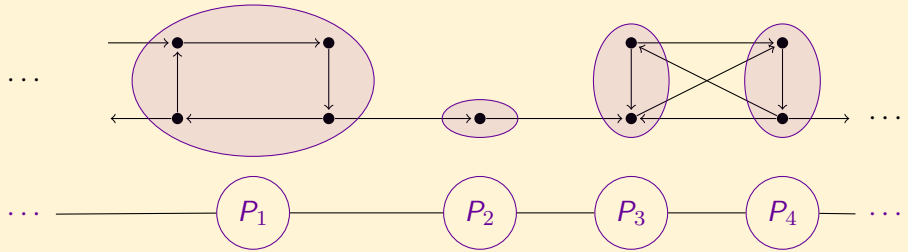
- General context-free graphs are tree-like:

Example



Thank you!