

Context-Free Graphs in Inverse Semigroup Theory

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Schützenberger Graphs

Let $M = \text{Inv}\langle A \mid \mathcal{R} \rangle$ be an inverse monoid,
let $A^{-1} = \{a^{-1} \mid a \in A\}$ be the disjoint set of **formal inverses** of A with $(a^{-1})^{-1} = a$ and
let $A^{\pm*} = (A \uplus A^{-1})^*$ be the set of words over A and A^{-1} .

The **Cayley graph** $\mathcal{C}(M)$ (for the generating set A) is the **directed** graph with
nodes M and **edges** $E = \{s \xrightarrow{a} sa \mid s \in M, a \in A^{\pm 1}\}$.

Definition (Schützenberger Graph)

The **Schützenberger graph** $SI(s)$ of $s \in M$ is the **strongly connected** component of $\mathcal{C}(M)$
containing s . *This is the \mathcal{R} -class of s !*

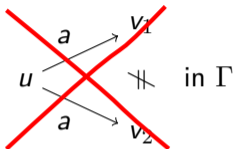
An inverse monoid is “fully determined” by its Schützenberger graphs.

Schützenberger graphs have many “desirable” properties of group Cayley graphs!

Properties of Schützenberger Graphs

Any Schützenberger graph Γ is...

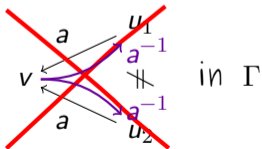
- deterministic:



- involutive:

$$u \xrightarrow{a} v \text{ in } \Gamma \iff u \xleftarrow{a^{-1}} v \text{ in } \Gamma$$

- co-deterministic: implied by the above!



- (strongly) connected

~~What can we use Schützenberger graphs for?~~

inverse graph

Intermezzo!

Intermezzo: Homomorphisms and Isomorphism of Directed Graphs

Consider directed $A^{\pm 1}$ -graphs $\Gamma = (V, E)$ and $\Delta = (W, F)$.

A **homomorphism** $\varphi: \Gamma \rightarrow \Delta$ is a map $V \rightarrow W$ with

$$v_1 \xrightarrow{a} v_2 \in E \implies \varphi(v_1) \xrightarrow{a} \varphi(v_2) \in F \quad \text{for all } a \in A^{\pm 1}.$$

An **isomorphism** $\varphi: \Gamma \rightarrow \Delta$ is a **bijection** $V \rightarrow W$ with

$$v_1 \xrightarrow{a} v_2 \in E \iff \varphi(v_1) \xrightarrow{a} \varphi(v_2) \in F \quad \text{for all } a \in A^{\pm 1}.$$

An **automorphism** of Γ is an isomorphism $\Gamma \rightarrow \Gamma$.

Now: What can we use Schützenberger graphs for?

Schützenberger Graphs, Maximal Subgroups and Green's Relations

- $\text{Aut } S\Gamma(s)$ is isomorphic to the maximal subgroup of M containing ss^{-1} . (Stephen, 1990)
 - $s \mathcal{R} t \iff$ there is an isomorphism $S\Gamma(s) \rightarrow S\Gamma(t)$ mapping $ss^{-1} \mapsto tt^{-1} \iff ss^{-1} = tt^{-1}$
 - $s \mathcal{L} t \iff$ there is an isomorphism $S\Gamma(s) \rightarrow S\Gamma(t)$ mapping $s \mapsto t \iff s^{-1}s = t^{-1}t$
 - $s \mathcal{D} t \iff$ there is an isomorphism $S\Gamma(s) \rightarrow S\Gamma(t)$
 - $s \geq_{\mathcal{J}} t \iff MsM \supseteq MtM$
 \iff there is a homomorphism $S\Gamma(s) \rightarrow S\Gamma(t)$
- We could use the word problem...

If we want to decide these algorithmically, we need to

- 1 compute/describe $S\Gamma(s)$ and
- 2 decide homomorphism/isomorphism problems.

We may use Stephen's procedure to "compute" $S\Gamma(s)$. But: this quickly yields undecidability!

What else can we do?

Schützenberger Graphs as Automata

Definition

A **deterministic automaton** \mathcal{A} is a **inverse $A^{\pm 1}$ -graph** with some node q_0 marked as **initial** and possibly multiple **final** nodes F .

It **accepts the language** $\mathcal{L}(\mathcal{A}) = \{w \in A^{\pm*} \mid q_0 \xrightarrow{w} f \in F\}$.

A **language** is a subset of $A^{\pm*}$.

Every **automaton** accepts a **unique language** and every **language** $L \subseteq A^{\pm*}$ has a **unique minimal automaton**.

Theorem (Stephen, 1990)

$S\Gamma(s)$ with **initial node** ss^{-1} and **final node** s is the **minimal automaton** for the language

$$\mathcal{L}(S\Gamma(s)) = \mathcal{U}(s) = \{u \in A^{\pm*} \mid u \geq s \text{ in } M\}.$$

Thus: $\mathcal{L}(S\Gamma(s)) = \mathcal{U}(s)$ fully describes $S\Gamma(s)$! But: undecidable in general

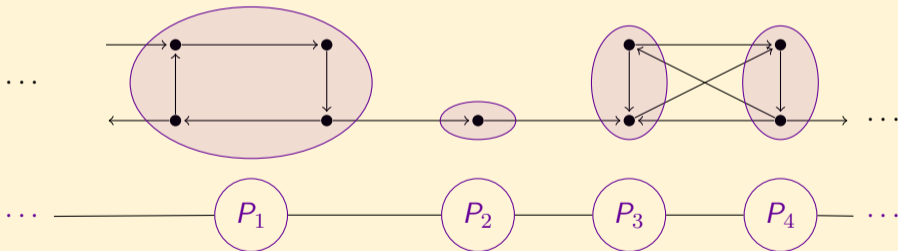
Tree-Like Graphs

A directed graph is **tree-like**

\iff it is **quasi-isometric** to a tree

\iff it has a **strong tree decomposition**

Example



Tree-Like Inverse Monoids

Theorem (Gray, Silva, Szakács, 2022)

Let $M = \text{Inv}\langle A \mid \ell_1 = r_1, \dots, \ell_n = r_n \rangle$ be *finitely presented* and $s \in M$. Then:
 $ST(s)$ is *tree-like* $\implies \mathcal{L}(ST(s))$ is *context-free*

In fact: This is true for certain tree-like *subgraphs* of Schützenberger graphs as well!

Used, for example, in:

Theorem (Gray, Kambites, W., WIP)

G, H : *finitely presented groups* Then:
There is a *finitely presented special inverse monoid* with

- G as *group of units* and
- H as the *maximal subgroup* of some idempotent.

What is a context-free language?

Context-Free Languages

For us:

$L \subseteq A^*$ is context-free $\iff L = \mathcal{L}(\mathcal{C})$ for some pushdown automaton \mathcal{C}

Definition

A pushdown automaton $\mathcal{C} = (Q, A, \Gamma, \delta, q_0, \perp, F)$ consists of:

- Q : finite set of states
- A : input alphabet
- Γ : stack alphabet
- $\delta \subseteq Q \times (A \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$: transition relation with elements $\tau \in \delta$ written as $\tau = p, a_0, A \rightarrow q, \gamma$
- $q_0 \in Q$: initial state
- $\perp \in \Gamma$: stack bottom symbol
- $F \subseteq Q$: final states

Configuration: $w \in A^*$, $p \in Q$, $\gamma' \in \Gamma^*$

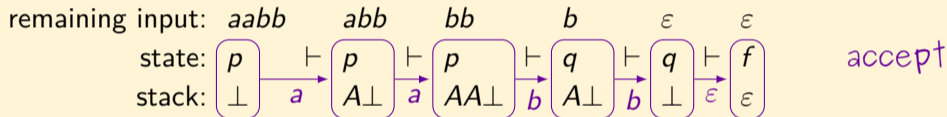
remaining input word:	$a_0 w$		w
state:	p	\vdash_τ	q
stack:	$A\gamma'$		$\gamma\gamma'$

Example PDA

Example

states: p (initial), q , f (final) **alphabet:** $\{a, b\}$ **stack symbols:** A, \perp (bottom symbol)

transitions: $p, a, \perp \rightarrow p, A\perp$ $p, a, A \rightarrow p, AA$ $p, b, A \rightarrow q, \varepsilon$
 $q, b, A \rightarrow q, \varepsilon$ $q, \varepsilon, \perp \rightarrow f, \varepsilon$



$$\mathcal{L}(\mathcal{C}) = \left\{ w \in A^* \mid \begin{pmatrix} w \\ q_0 \\ \perp \end{pmatrix} \vdash^* \begin{pmatrix} \varepsilon \\ f \\ \gamma \end{pmatrix} \text{ for } f \in F, \gamma \in \Gamma^* \right\}$$

$$= \{ a^n b^n \mid n \geq 1 \}$$

\rightsquigarrow Configuration graph $\Gamma(\mathcal{C})$ with edges $\{(p, A\gamma') \xrightarrow{a_0} (q, \gamma\gamma') \mid (p, a_0, A \rightarrow q, \gamma) \in \delta\}$

Context-Free Graphs

Γ : inverse $A^{\pm*}$ -graph with root u Then:

Some details on ε -transitions and \perp omitted

Γ is context-free $\iff \Gamma$ has finitely many end-isomorphism classes (definition)

$\iff \Gamma$ is the configuration graph of some PDA (Muller, Schupp, 1990)

$\iff \mathcal{L}(\Gamma; u, u)$ is deterministic context-free (Rodaro, 2024)

$\implies \Gamma$ is tree-like

" \Leftarrow " if Schützenberger graph of a f.p. inverse semigroup (Gray, Silva, Szakács)

Now: We may use deterministic PDAs to describe Schützenberger graphs.

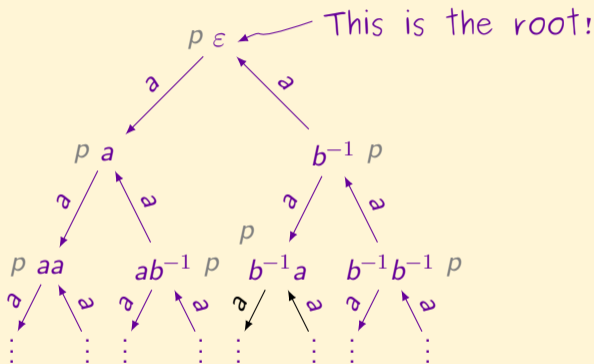
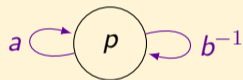
But: PDAs are algorithmically not "nice". Can we do better? *Probably...*

Definition (pDFA)

A **pDFA** is a finite, deterministic $A^{\pm 1}$ -graph whose nodes we call **states** and whose edges we call **transitions**.

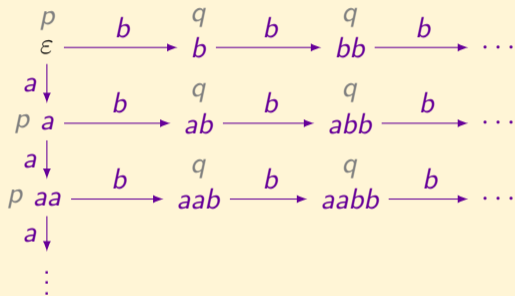
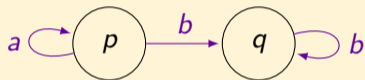
Every state p of a pDFA generates an **involution** tree via its tree of runs/path space.

Example



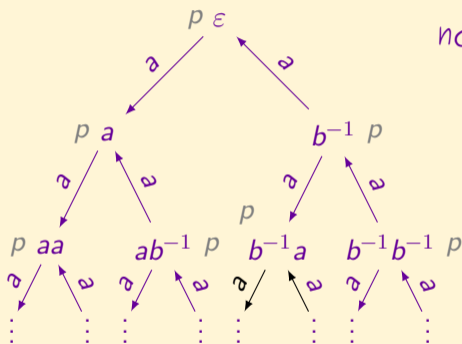
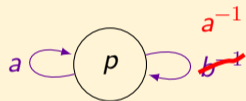
$\Gamma(p)$

Example



$\Gamma(p)$

Example



not deterministic!

$\Gamma(p)$

$\Gamma(p)$ is inverse and deterministic $\iff aa^{-1}$ cannot be read for any $a \in A^{\pm 1}$
 \iff the pDFA is reduced

Γ : inverse $A^{\pm 1}$ -tree, i. e. a tree as an unlabeled, undirected graph

Theorem (W., arXiv 2026)

Γ is *context-free* $\iff \Gamma$ arises as $\Gamma(p)$ for a state p of a *reduced pDFA*

Context-free trees

- arise naturally as **Schützenberger graphs** of
 - free inverse monoids (**Munn trees**).
 - inverse monoids with idempotent relators. \rightsquigarrow *more on next slide*
- hopefully serve as a **stepping stone** for **general context-free graphs**.

Lemma (Margolis, Meakin, 1993)

$$M = \text{Inv}\langle A \mid \ell_i = r_i, i \in I \rangle \text{ for } \ell_i = 1 = r_i \text{ in } F(A)$$

\iff all Schützenberger graphs of M are trees and, thus, context-free if $|I| < \infty$

Theorem (Margolis, Meakin, 1993)

The word problem

Constant: $M = \text{Inv}\langle A \mid \ell_i = r_i \rangle$ for $\ell_i = 1 = r_i$ in $F(A)$

Input: $u, v \in A^{\pm*}$

Question: is $u = v$ in M ?

is decidable.

This solves \mathcal{R} and \mathcal{L} . What about \mathcal{D} and \mathcal{J} ?

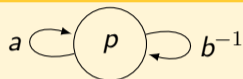
\rightsquigarrow Let's look at the homomorphism/

isomorphism problem of context-free trees!

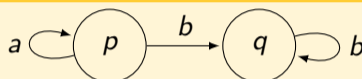
The Rooted Isomorphism Problem

$\mathcal{L}(p)$: words readable from state p in its pDFA

Example



$$\mathcal{L}(p) = \{a, b^{-1}\}^*$$



$$\mathcal{L}(p) = a^*b^*$$



$$\mathcal{L}(p) = \{a, a^{-1}\}^*$$

Fact

Let p and q be *states of reduced pDFAs*. Then:

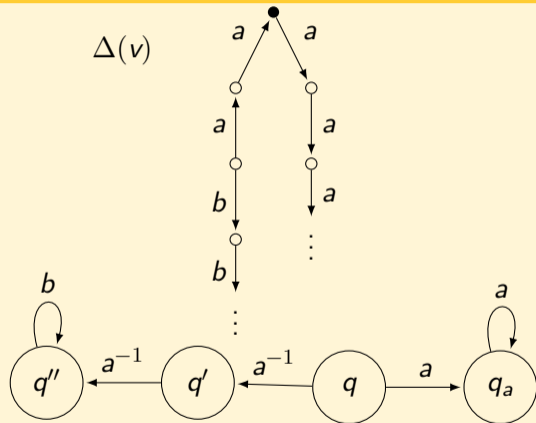
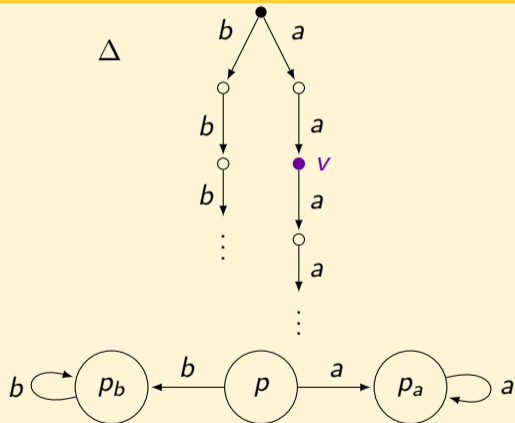
$$\Gamma(p) \cong \Gamma(q) \iff \mathcal{L}(p) = \mathcal{L}(q)$$

isomorphic as rooted graphs

It is well known that the latter is in NL and, thus, **polynomial time**.

The Non-Rooted Case: The Problem

Example



The languages don't seem connected!

The Non-Rooted Isomorphism Problem

Theorem (W., arXiv 2026)

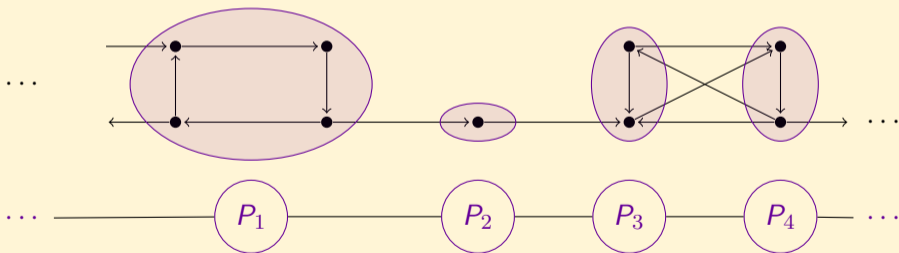
The non-rooted isomorphism problem for inverse context-free trees

Input: states \check{p} and \check{q} of *reduced pDFAs*

Question: are $\Gamma(\check{p})$ and $\Gamma(\check{q})$ isomorphic as *non-rooted graphs*
is in **NL** and, thus, in particular, decidable in deterministic *polynomial time*.

- We still need to **compute** a generating reduced **pDFA** for $ST(s)$ from s and the presentation.
- General context-free graphs are **tree-like**:

Example



Thank you!