Equations in Wreath Products of Abelian Groups

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joint work with
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Theorem (Matiyasevich; 1970)

The problem is undecidable.

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Where is the border between decidability and undecidability?

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It satisfies (is a solution of) a system $\{w_i = 1\}$ if $\sigma(w_i) = 1$ holds in G for all i.

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Definition (Diophantine Problem)

The Diophantine problem DP is the decision problem:

Constant: the group $G = \langle \Sigma \rangle$
Input: a finite system of equations $\{w_1 = 1, \ldots, w_\ell = 1\}$
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- DP is undecidable in $\mathbb{Z} \wr \mathbb{Z}$ (Dong 2024)

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This is a special form of DP_1 where the equation only contains constants.

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This is equivalent to asking whether $ZgZ^{-1} = h$ has a solution in G and, thus, a special form of DP_1 as well.

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For example…

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Then: $\sigma(w) = 1 \iff \sigma(X)^{\varepsilon} = \sigma(u^{-1}v^{-1})$ and we always have a solution. □

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In fact: The normal form can be efficiently computed.

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Proposition

The problem

Constant: any group $A \wr B$ for abelian groups A and B

Input: a nonorientable equation $\prod_{s=1}^{S} Y_s \prod_{k=1}^{K} Z_k c_k Z_k^{-1} = 1$ for $c_k \in A \wr B$

Question: does it have a solution?

is decidable.

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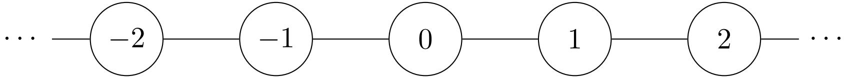
We will start with the classic lamplighter group $L_2 = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.

Elements of the Lamplighter Group

An element of the lamplighter group is represented by

Elements of the Lamplighter Group



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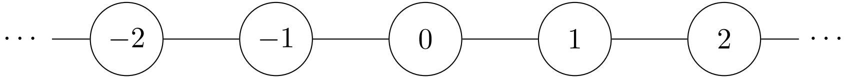

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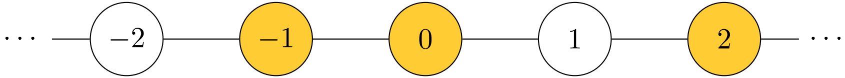

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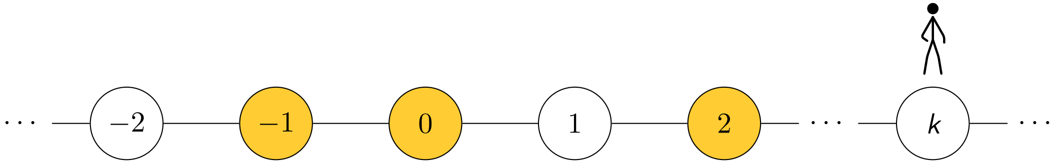

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- the location of the lamplighter.

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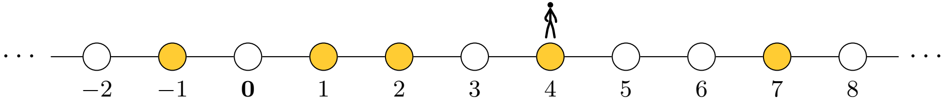

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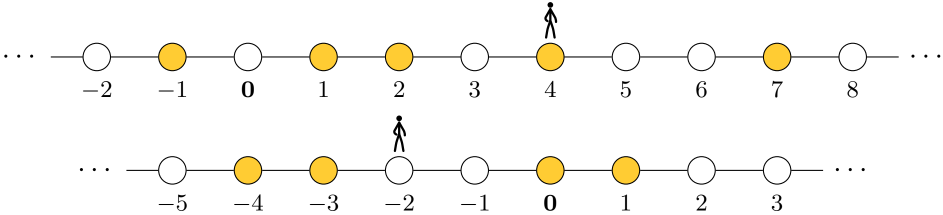

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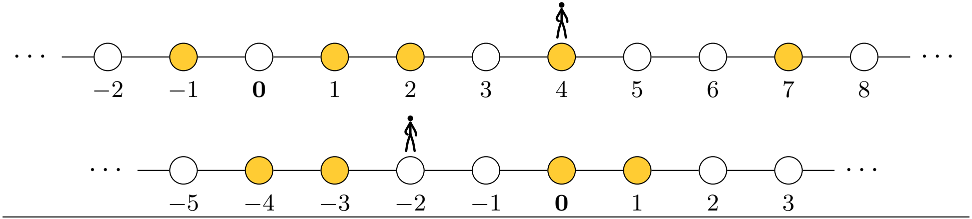

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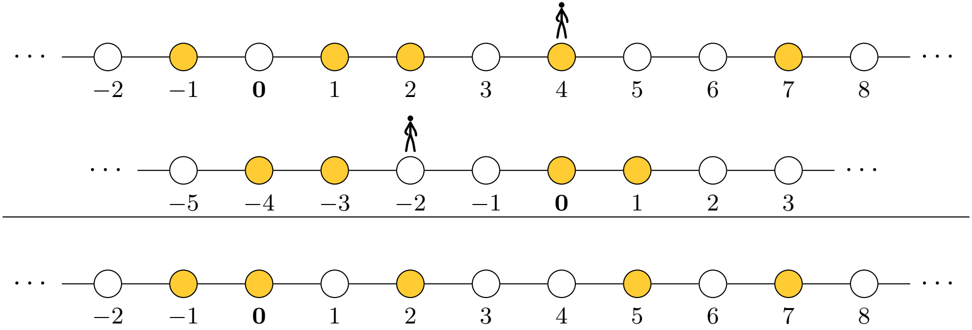

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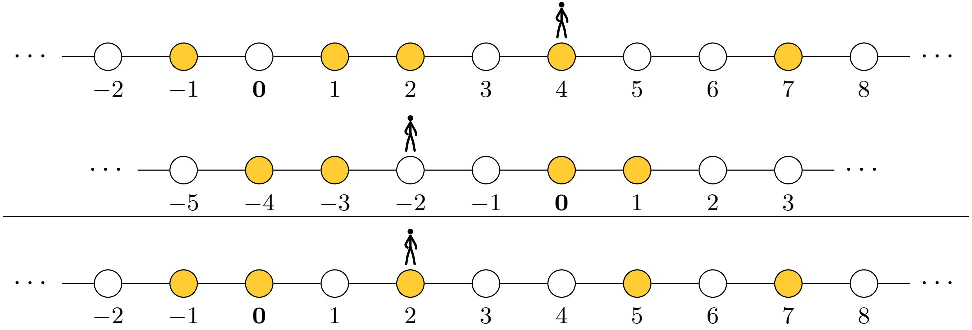

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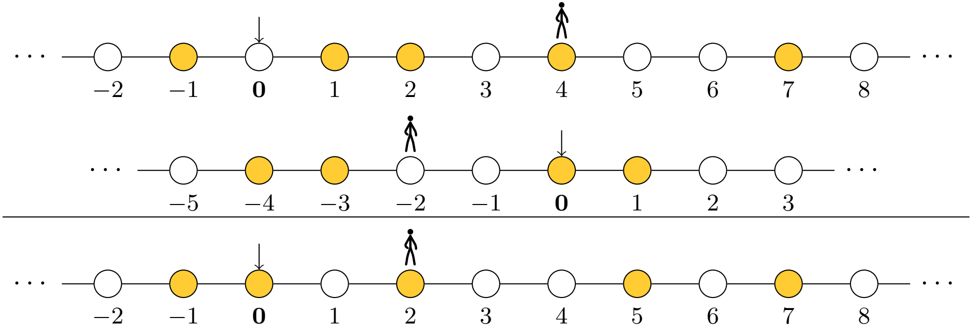

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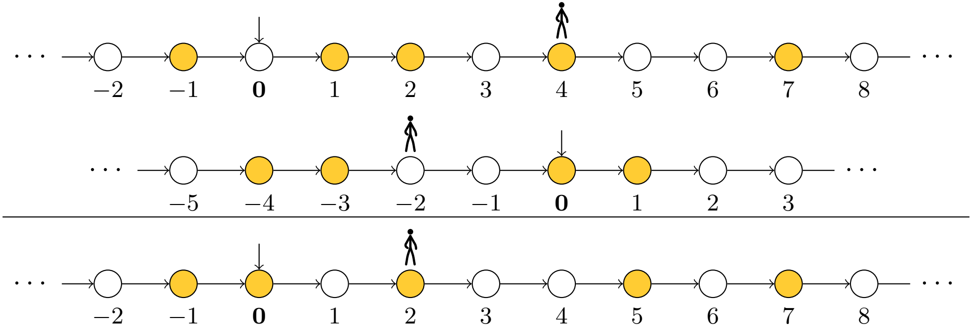

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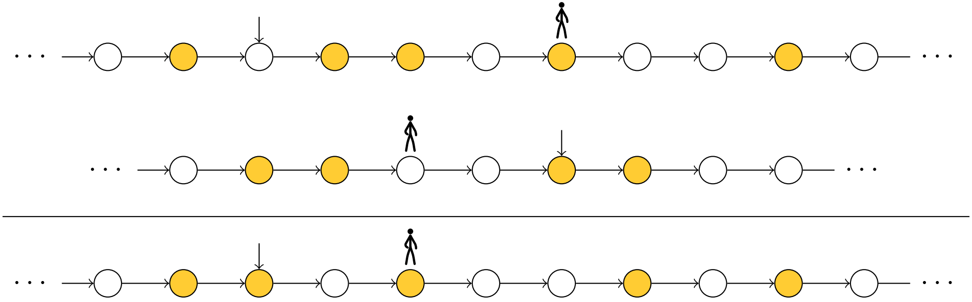

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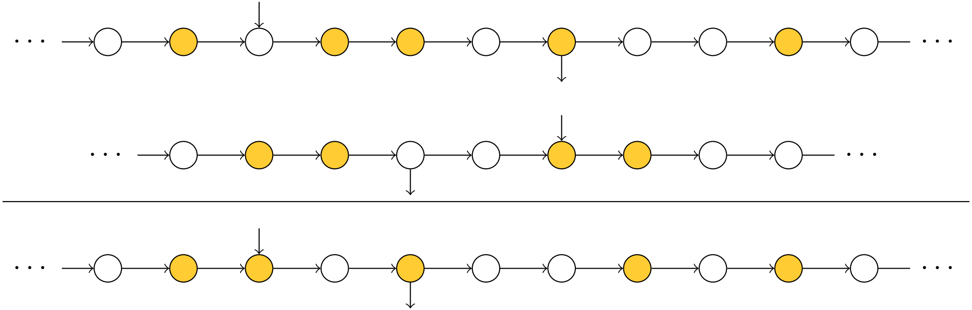

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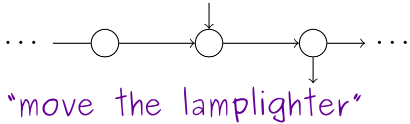
$t =$ 

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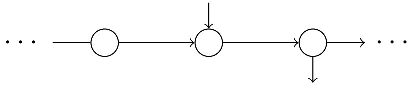
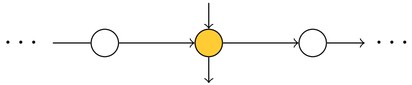
$t =$ 

"move the lamplighter"

$a =$ 

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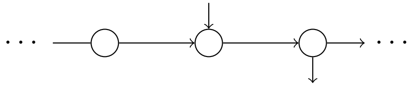
$t =$ 

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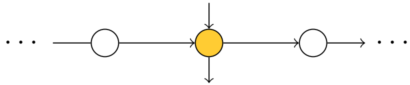
$a =$ 

"toggle the current lamp"

In fact: $L_2 = \langle a, t \mid a^2 = 1, [a, t^\ell a t^{-\ell}] = 1, \ell \in \mathbb{Z} \rangle$.

Conjugacy in the Lamplighter Group

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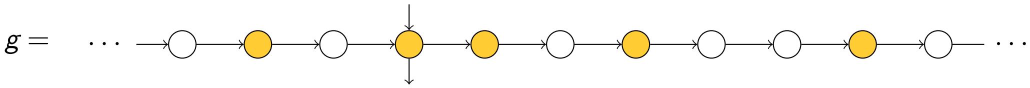

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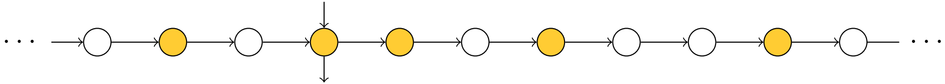

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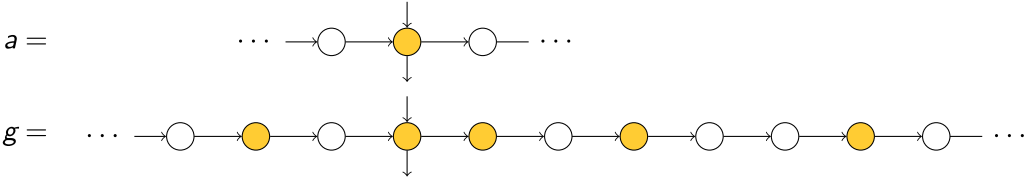

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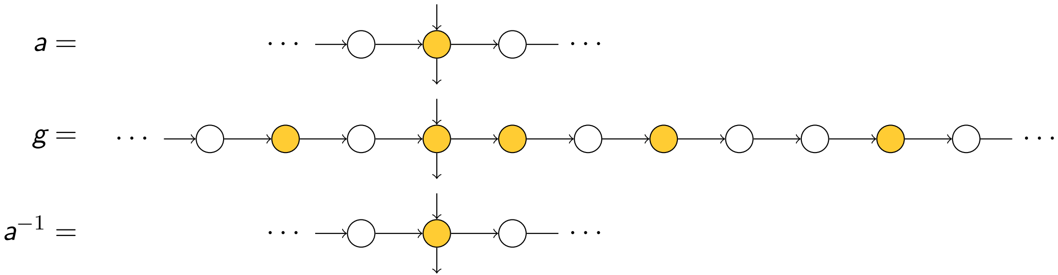

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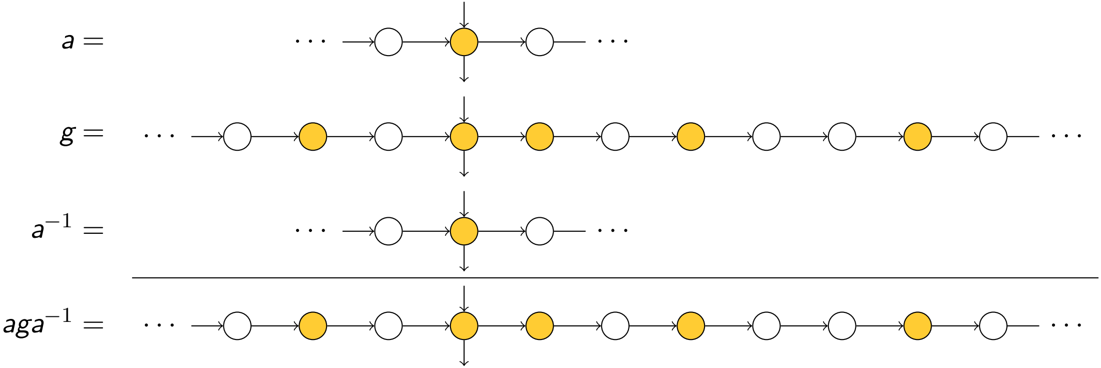

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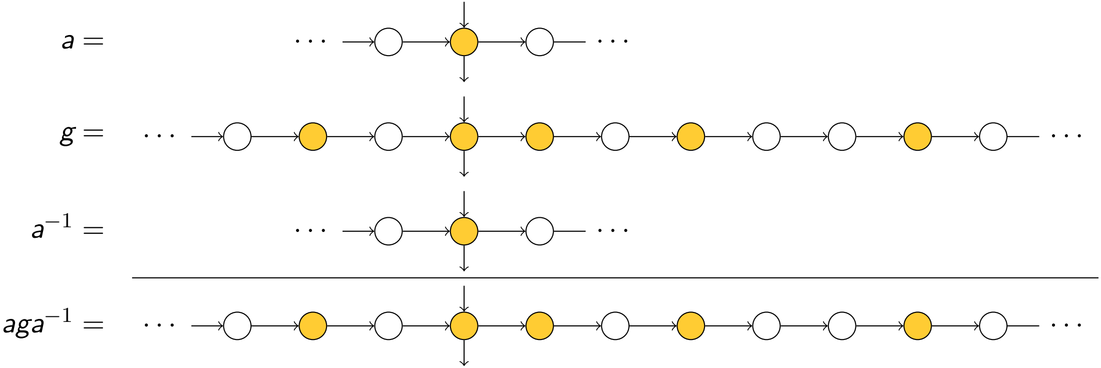

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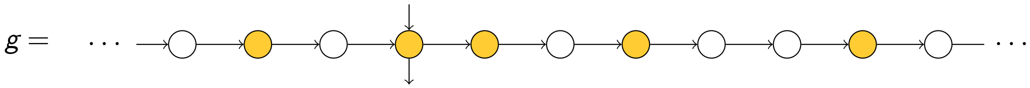

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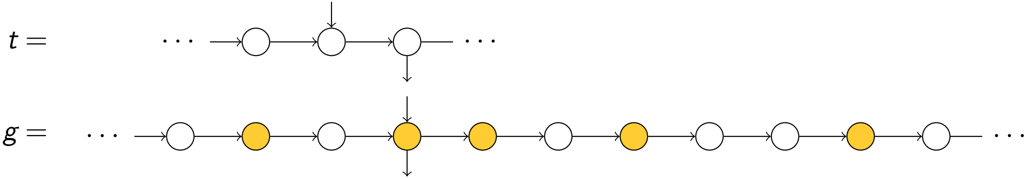

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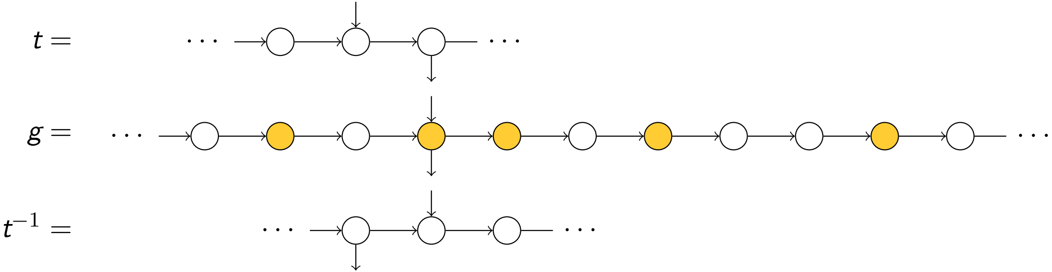

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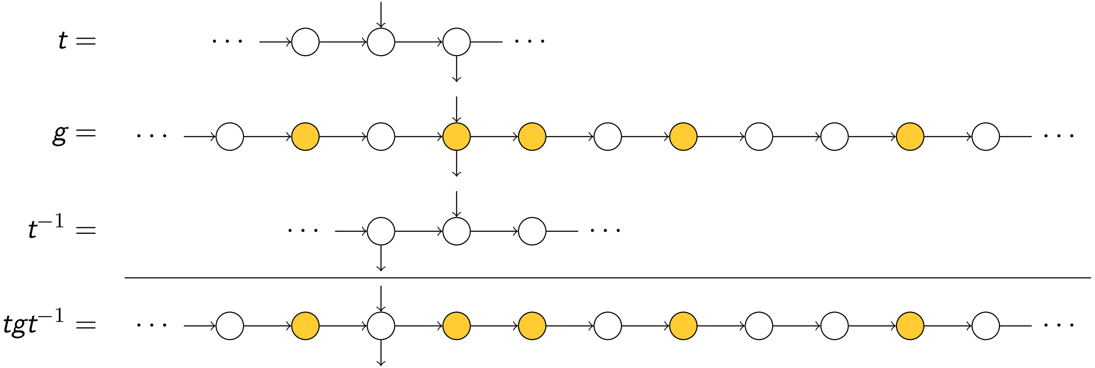

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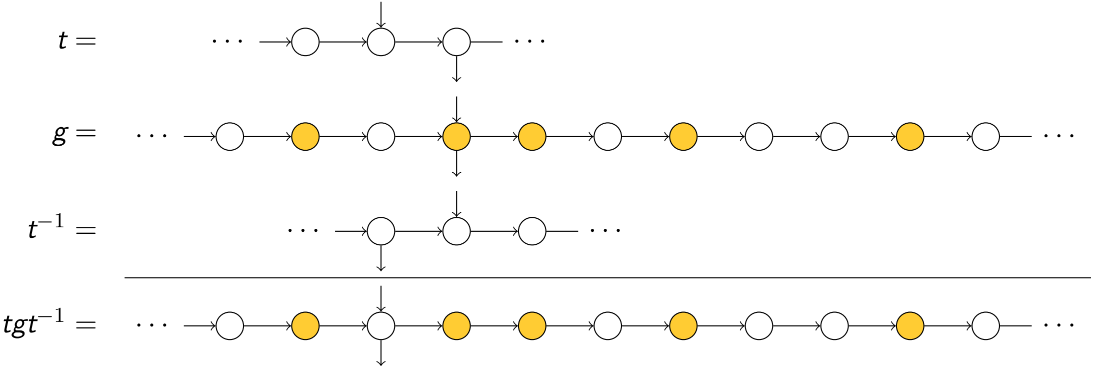
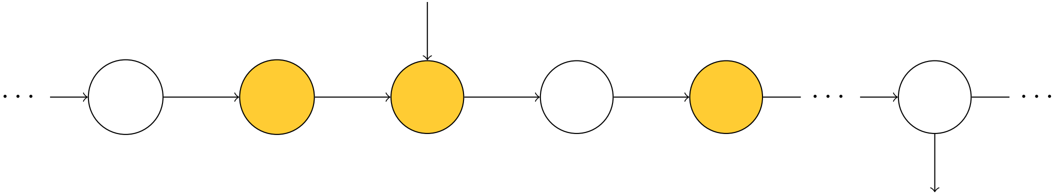


Generalized Lamplighter Groups



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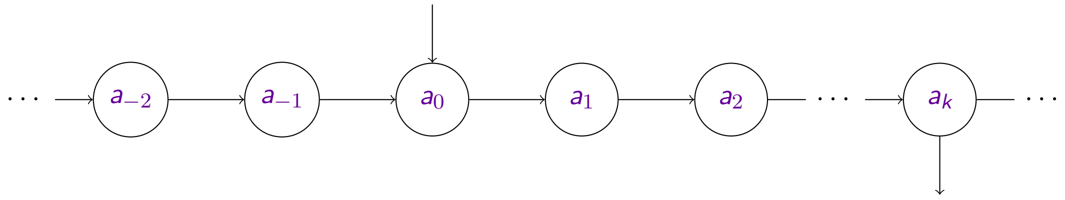

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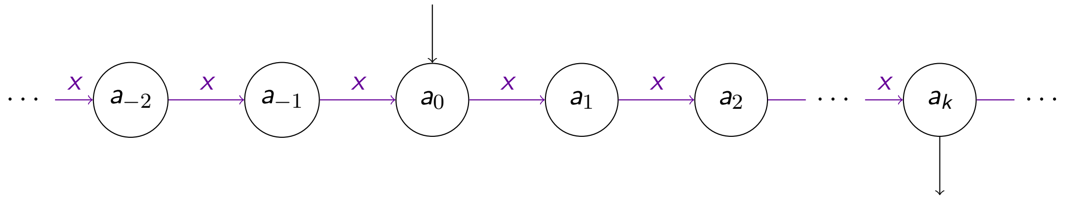

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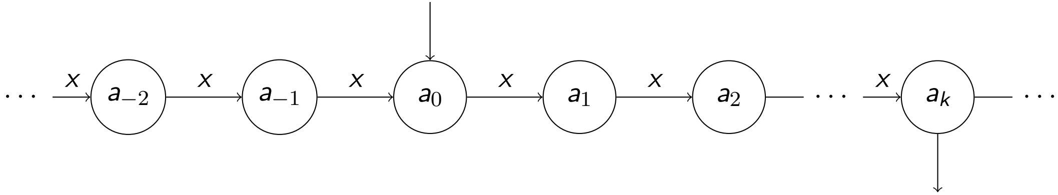

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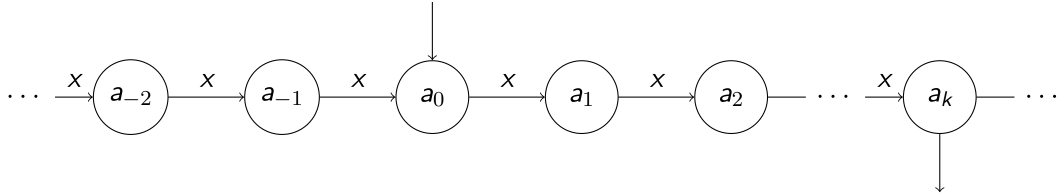

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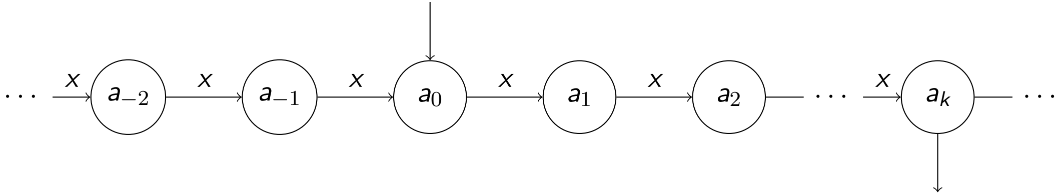

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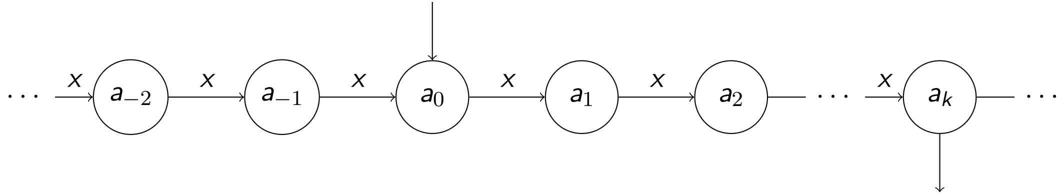

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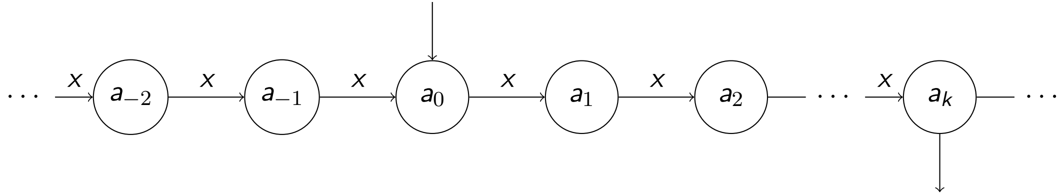

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We obtain: functions $B \to A$ with finite support and an element of B as the lamplighter position

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This allows us to define the ring of Laurent polynomials in multiple variables over A...

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The set of all Laurent polynomials is $A[\boldsymbol{X}^{\pm 1}]$.

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Abelian Groups and Lattices

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A lattice is a finitely generated additive subgroup of $\mathbb{Z}^d$

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A lattice is a finitely generated additive subgroup of $\mathbb{Z}^d$

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Of course: We may view a lattice as an extended lattice!

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Acting on Laurent Polynomials

The additive group $\mathbb{Z}^d$ acts on $A[\boldsymbol{X}^{\pm 1}]$:
$\boldsymbol{v} \in \mathbb{Z}^d$ acts on $f \in A[\boldsymbol{X}^{\pm 1}]$ by $\boldsymbol{X}^{\boldsymbol{v}} f$

Acting on Laurent Polynomials

Recall: $\mathscr{I}(L) = \langle \boldsymbol{X}^{\ell} = \mathbb{1} \mid \ell \in L \rangle$

The additive group $\mathbb{Z}^d / L = B$ acts on $A[\boldsymbol{X}^{\pm 1}] / \mathscr{I}(L)$:

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NOW: $B = \mathbb{Z}^d / L$ acts on $A[\boldsymbol{X}^{\pm 1}] / I$ for $I = \mathscr{I}(L)$ as a group

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- Note: g gets shifted by $\boldsymbol{X}^{\boldsymbol{b}}$.

Solving Nonorientable Equations

Magic Lemma 1

$g_k \in A^{(B)}$, $\operatorname{supp} g_k \subseteq [-D, D]^d$, $\boldsymbol{m}_k \in \mathbb{Z}^d$

$$\prod_{s=1}^{S} Y_s^2 \prod_{k=1}^{K} Z_k(g_k, \boldsymbol{m}_k) Z_k^{-1} = 1$$ has a solution in $A \wr B$

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Idea: $\quad (f, \boldsymbol{n})^2$

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$I \supseteq \mathscr{I}(L)$: *ideal of* $A[\boldsymbol{X}^{\pm 1}]$, $\quad w \in (A \wr B) \star F(\mathbb{X} \setminus \{Y\})$

Then: $\quad Y^2 w = (\mathbb{0}, \mathbf{0})$ *has a solution in* $A[\boldsymbol{X}^{\pm 1}]/I \rtimes \mathbb{Z}^d/L$

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Idea: $\quad (f, \boldsymbol{n})^2 = (f, \boldsymbol{n})\,(f, \boldsymbol{n}) = (f + \boldsymbol{X}^{\boldsymbol{n}} f, 2\boldsymbol{n}) = \big(f(\mathbb{1} + \boldsymbol{X}^{\boldsymbol{n}}), 2\boldsymbol{n}\big)$

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$$g_k \in A[\boldsymbol{X}^{\pm 1}],\ \operatorname{supp} g_k \subseteq [-D, D]^d, \qquad \boldsymbol{m}_k \in \mathbb{Z}^d$$

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Result After Magic Lemma 2

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$\boldsymbol{X}^{\boldsymbol{m}_k} = \mathbb{1}$ by Magic Lemma 4

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Recall: "Lamplighter at origin $\implies$ conjugation is translation"

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$$\iff \exists \boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_K \in \mathbb{Z}^d : \sum_{k=1}^{K} \boldsymbol{X}^{\boldsymbol{\kappa}_k} g_k = \mathbb{0} \text{ in } A[\boldsymbol{X}^{\pm 1}]/I$$

(I — any ideal)

Some Combinatorics

$\exists \boldsymbol{n}'_1, \ldots, \boldsymbol{n}'_{S-1} \exists \boldsymbol{n}$: ❶ $\sum_{k=1}^{K} \boldsymbol{m}_k = 2\boldsymbol{n}$ in $\mathbb{Z}^d / L$ and

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Observation: We may move along the lattice $\langle L, \boldsymbol{n}'_s, \boldsymbol{n}, \boldsymbol{m}_k \rangle$ to make the $\boldsymbol{\kappa}_k$ "small"

Some Combinatorics

$\exists \boldsymbol{n}'_1, \dots, \boldsymbol{n}'_{S-1} \exists \boldsymbol{n}:$ ❶ $\sum_{k=1}^{K} \boldsymbol{m}_k = 2\boldsymbol{n}$ in $\mathbb{Z}^d / L$ and

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$$\prod_{s=1}^{S} Y_s^2 \prod_{k=1}^{K} Z_k(g_k, \boldsymbol{m}_k) Z_k^{-1} = (\mathbb{0}, \mathbf{0}) \text{ has a solution in } A \wr B$$

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We may treat everything except the $\boldsymbol{n}'_s$ as constants!

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$$\exists \boldsymbol{n}'_1, \ldots, \boldsymbol{n}'_{S-1} : \sum_{k=1}^{K} \sigma_k \boldsymbol{X}^{\boldsymbol{\kappa}'_k} g_k = \mathbb{0} \text{ in } A[\boldsymbol{X}^{\pm 1}]/(\langle \boldsymbol{X}^{\boldsymbol{L}} = (-\mathbb{1})^{S+1} \boldsymbol{X}^{\boldsymbol{n}} = \boldsymbol{X}^{\boldsymbol{m}_k} = \mathbb{1} \rangle + \langle \boldsymbol{X}^{\boldsymbol{n}'_s} = -\mathbb{1} \rangle)$$

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Proposition

The problem

Input:

Question:

is decidable.

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Proposition

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Input: $f \in A[\boldsymbol{X}^{\pm 1}]$,
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Thank you!