MANCHESTER 1824
The University of Manchester

The Freeness Problem for Automaton Semigroups

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joint work with
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12 March 2026

Algorithmic Problems in Group Theory

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Definition (Isomorphism Problem)

Input: two groups G and H
Question: are G and H isomorphic?

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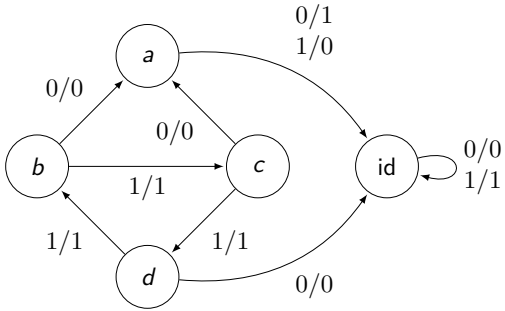

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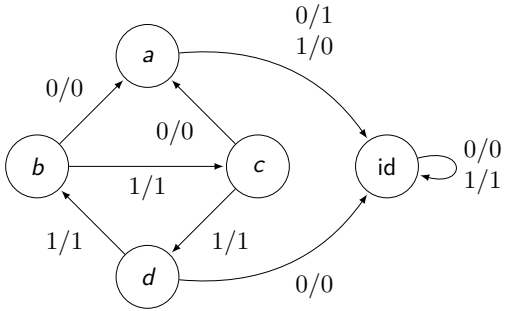

"How does this work?"
⇝ Later!

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These proofs are all somewhat ad-hoc...

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We will discuss the slightly simpler monoid case.

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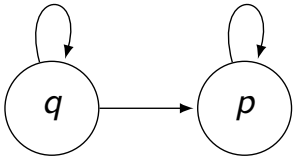

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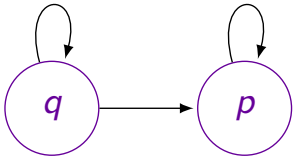

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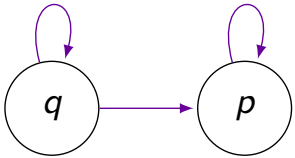

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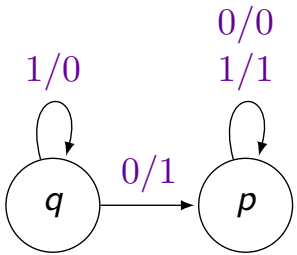

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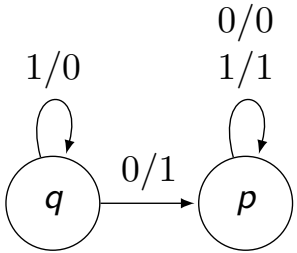

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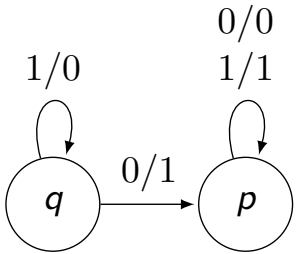

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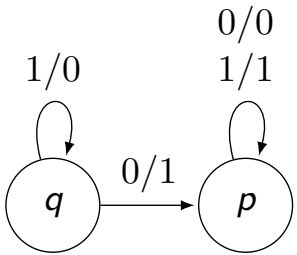

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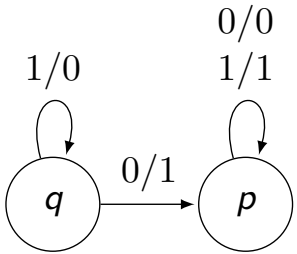

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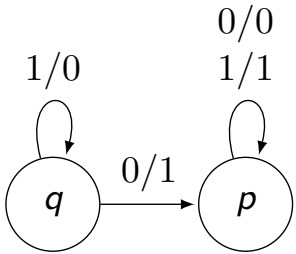

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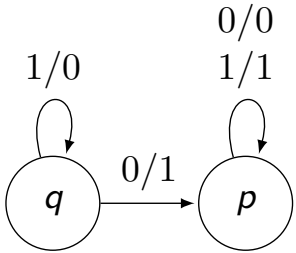

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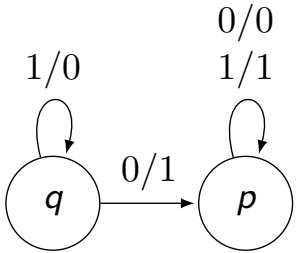

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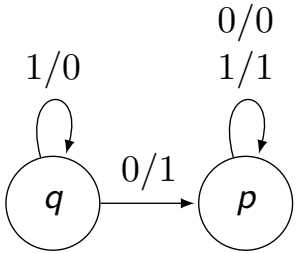

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Today we only consider deterministic and complete automata!

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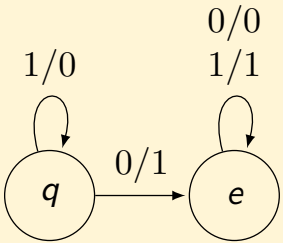
- Idea: every state sequence $\boldsymbol{q} \in Q^+$ induces an action $\Sigma^* \to \Sigma^*, u \mapsto \boldsymbol{q} \circ u$ mapping input words to output words.

Example



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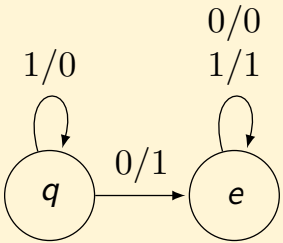

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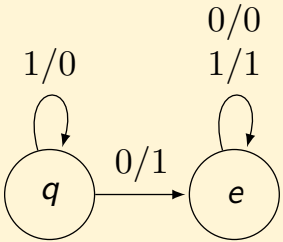

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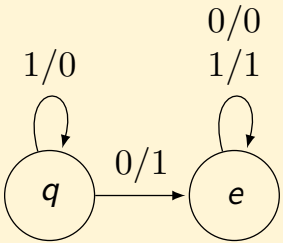

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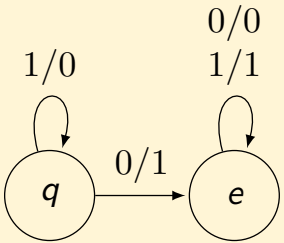

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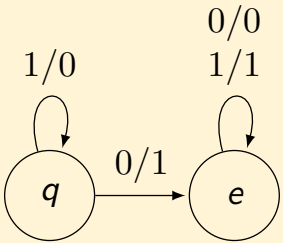

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$\rightsquigarrow$ the action of q increments (reverse) binary representation (least significant bit first)

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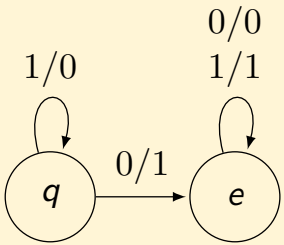

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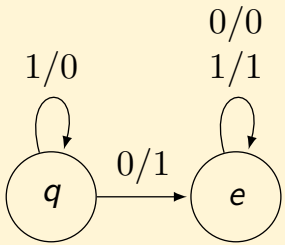

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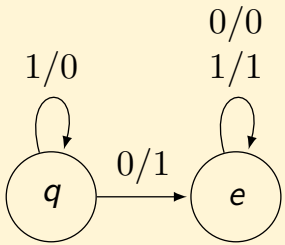

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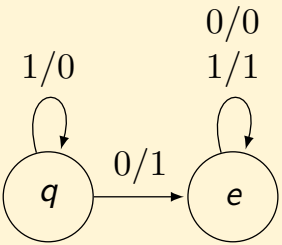

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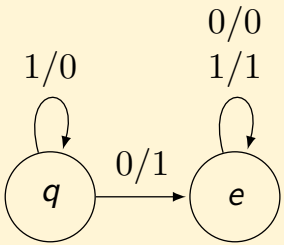

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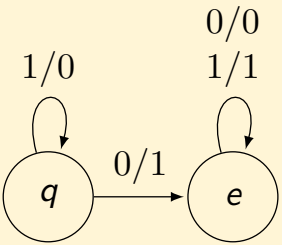

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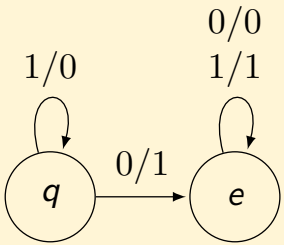

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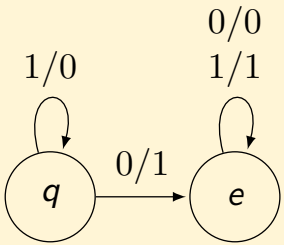

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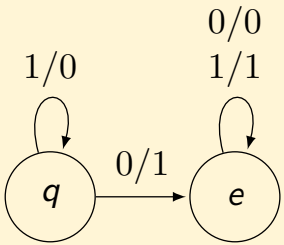

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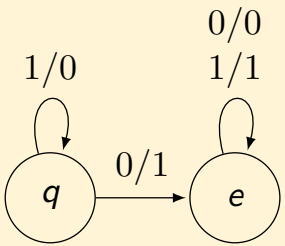

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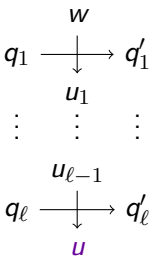

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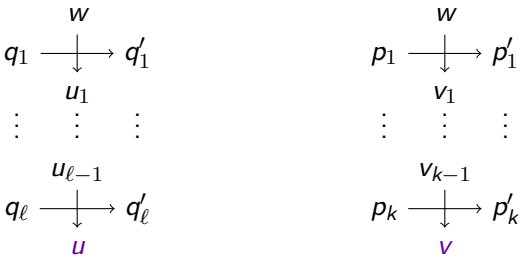

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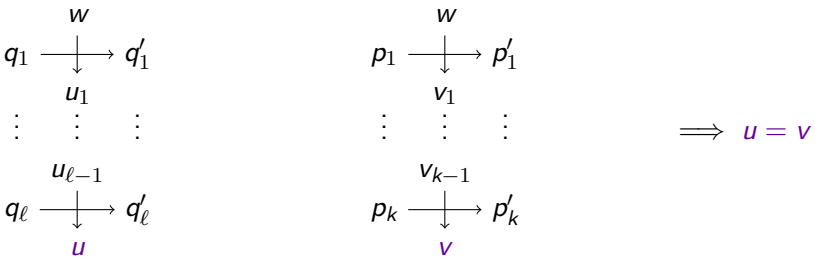

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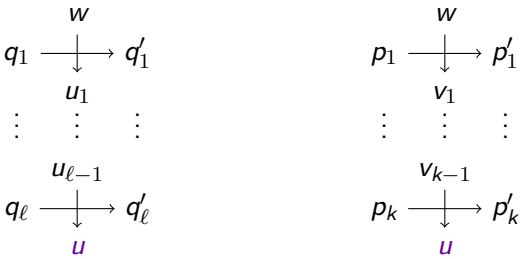

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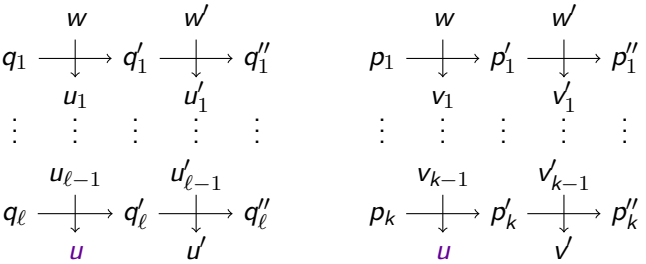

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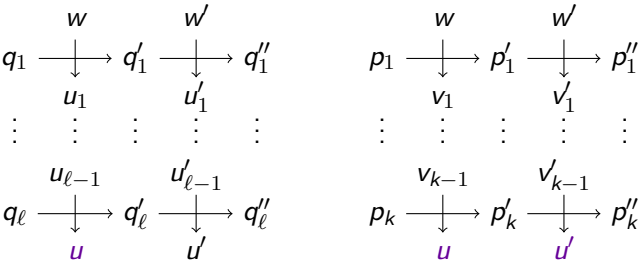

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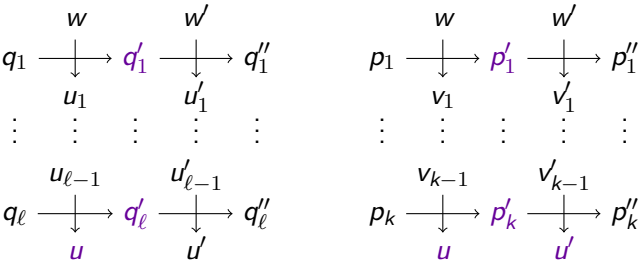

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Idea: Swap letters and states

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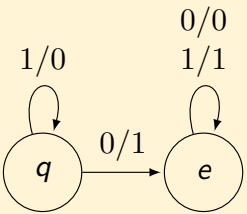

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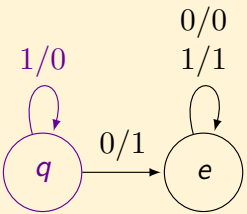

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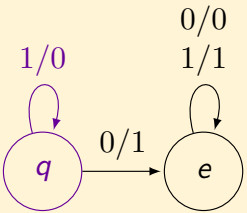

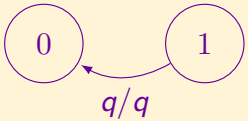

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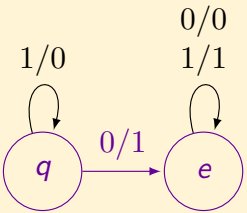

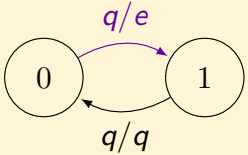

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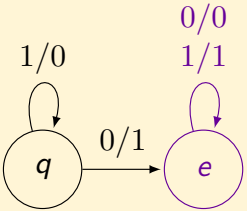

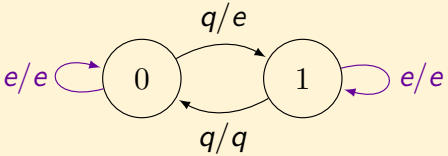

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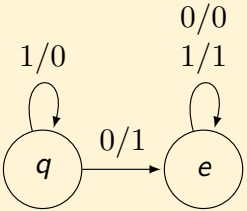

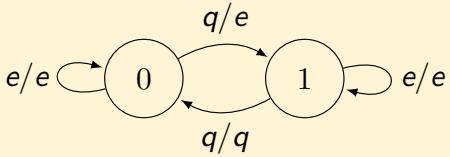

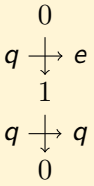

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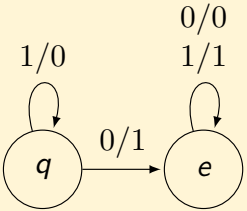

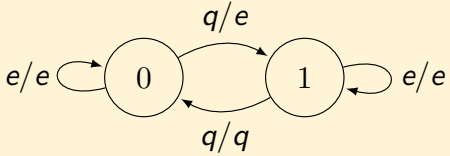

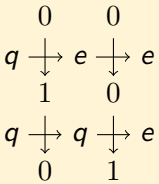

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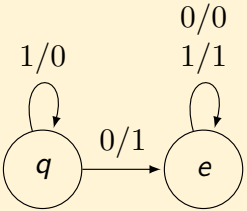

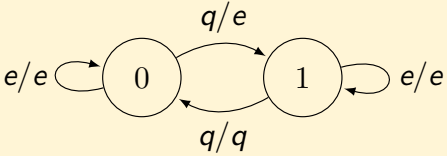

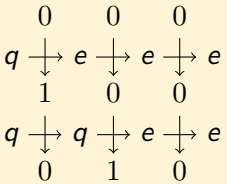

The Freeness Problem

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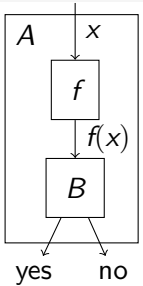

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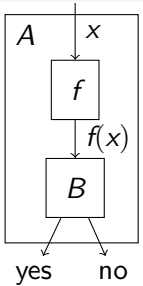

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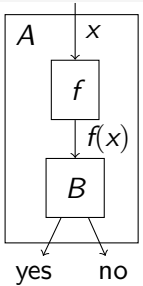

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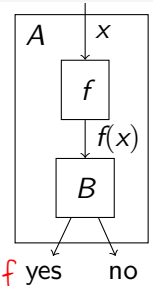

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maps $\varphi, \psi \colon I = \{1, \dots, n\} \to \Lambda^*$

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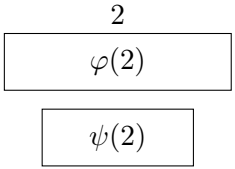

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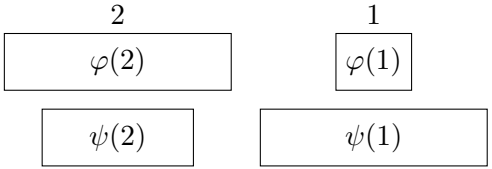

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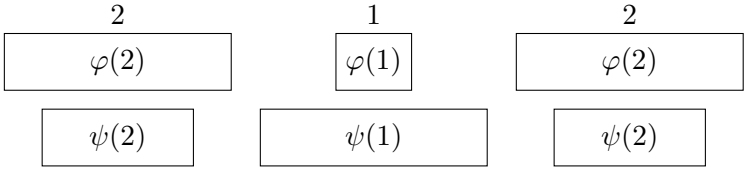

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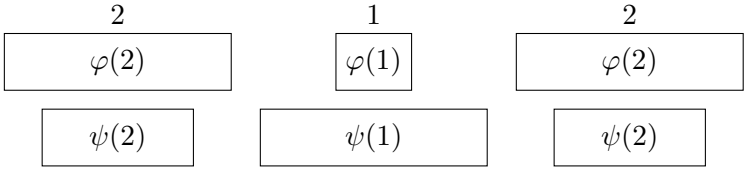
Question: $\exists \boldsymbol{i} \in I^+ : \varphi(\boldsymbol{i}) = \psi(\boldsymbol{i})$? (extended into hom.)

Idea: We have $n = |I|$ many tiles and must "match" the upper with the lower entries.

Example:



PCP is a classical undecidable problem!

Post's Correspondence Problem

Definition (Post's Correspondence Problem)

Constant: an alphabet Λ and a padding symbol $e \notin \Lambda$
Input: a number L and maps $\varphi, \psi \colon I = \{1, \dots, n\} \to (\Lambda \cup \{e\})^L$
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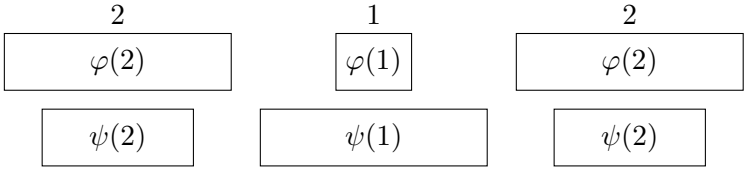

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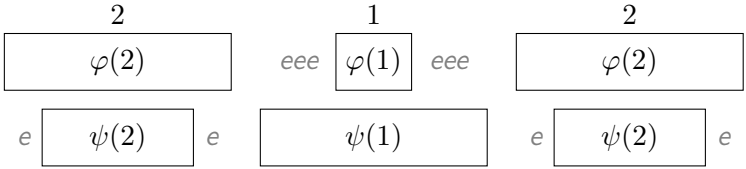

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Proof

The Reduction

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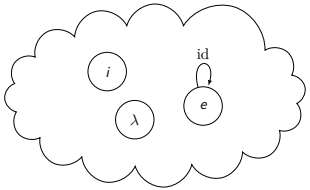

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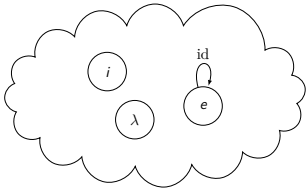
Relations: $u =_{\mathcal{T}} u$ for $u \in (\Lambda \cup I)^*$



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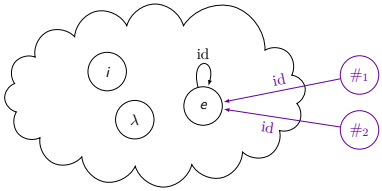
Relations:



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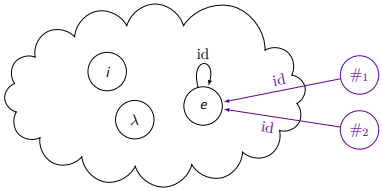
Relations: $u_0 \#_? u_1 \dots \#_? u_k =_\tau v_0 \#_? v_1 \dots \#_? v_\ell$

$$\iff u_0 \dots u_k = v_0 \dots v_\ell$$



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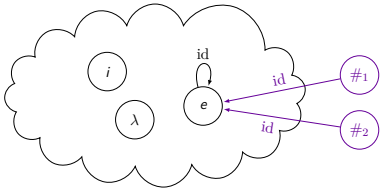
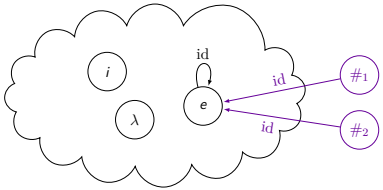
Now: ensure equality in the blocks...



Blocks in a Relation



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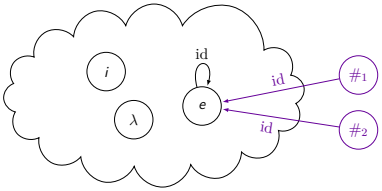

(dual automaton)

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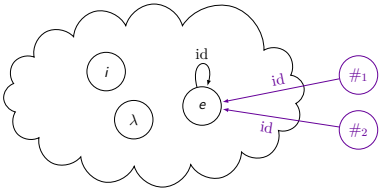

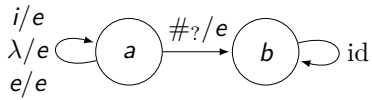

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Why? ⇝ It is a bit simpler & easier to understand.

Blocks in a Relation

u_0

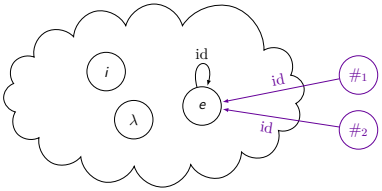
$\#_?$

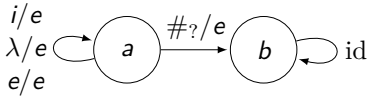
u_1

$\#_?$

$\#_?$

u_k





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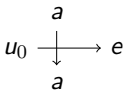


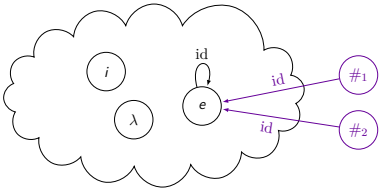
$\#_?$

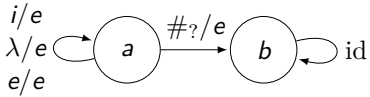
u_1

$\#_?$

$\#_?$

u_k





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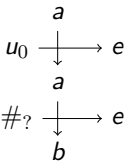

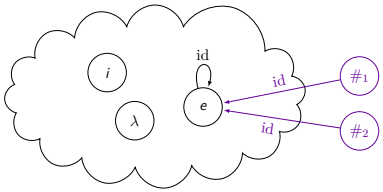

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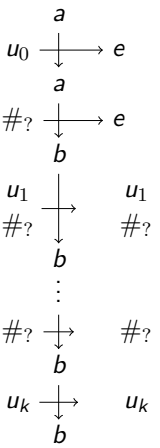

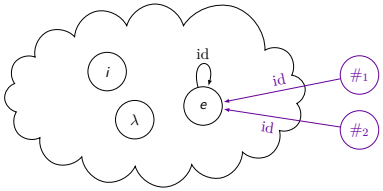

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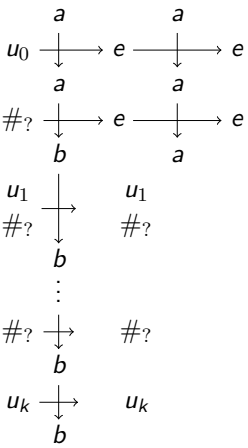

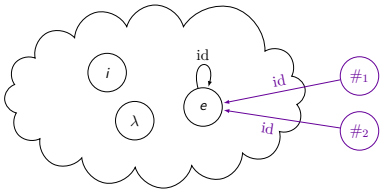

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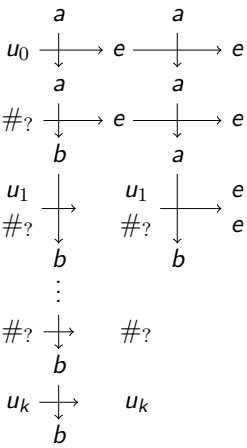

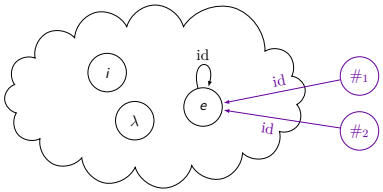

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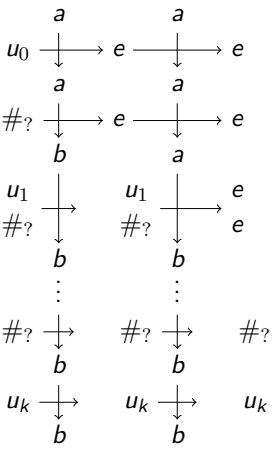

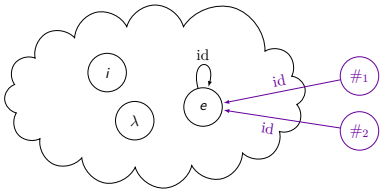

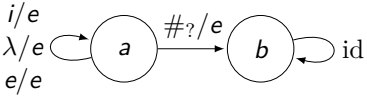

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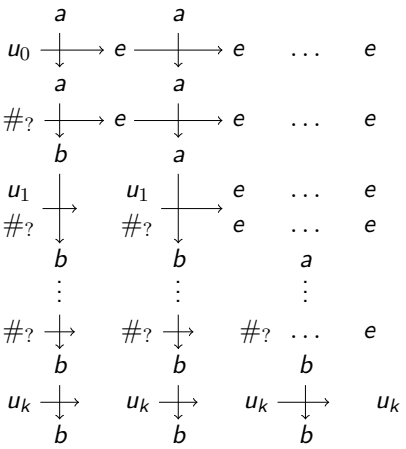

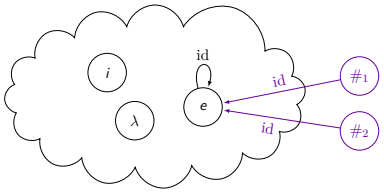

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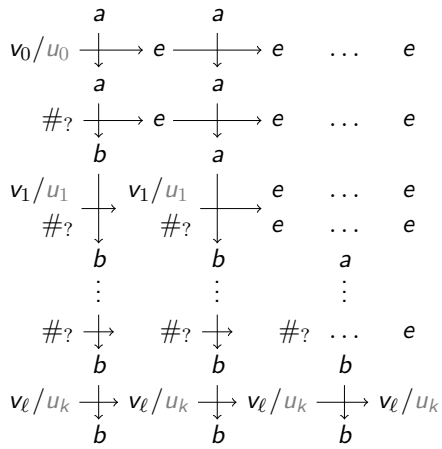

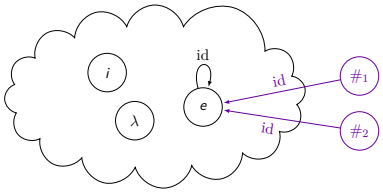

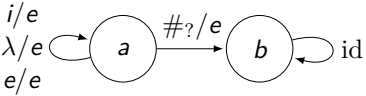

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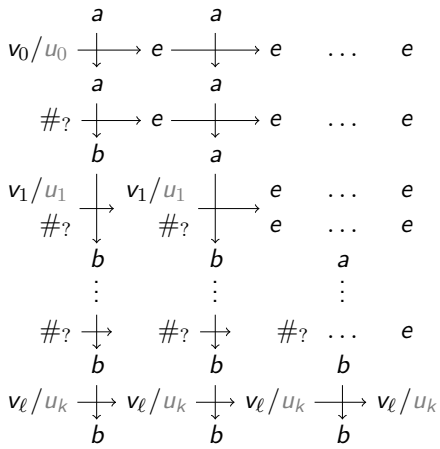
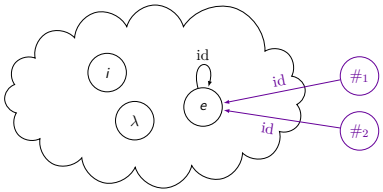


$k < \ell$



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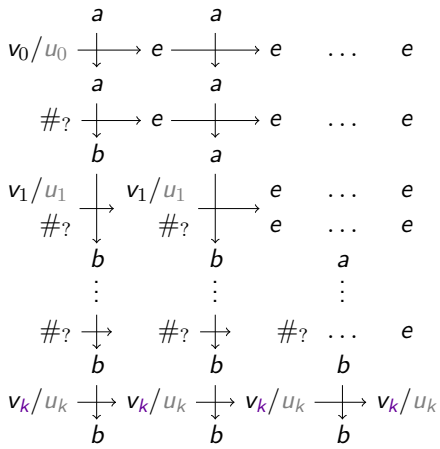
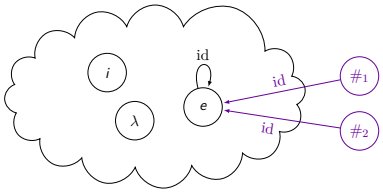
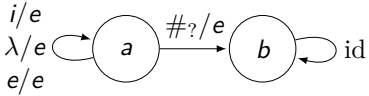


$\Longrightarrow \quad k = \ell$





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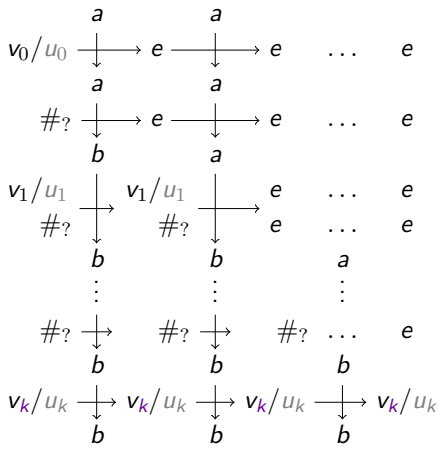
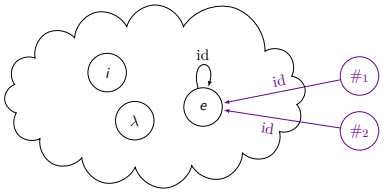
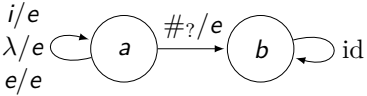


$\Longrightarrow$ $k = \ell$ and $u_i = v_i$





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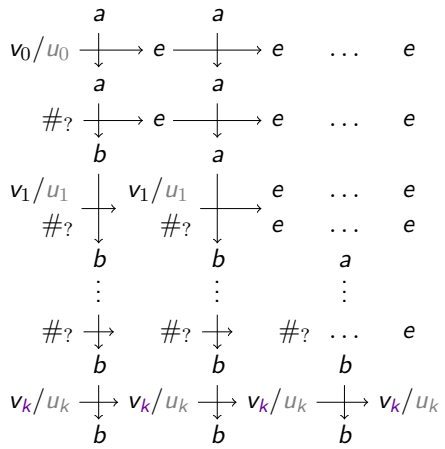


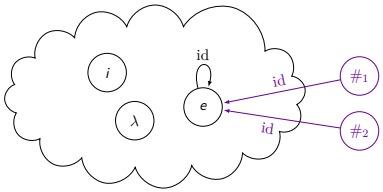
$\implies k = \ell$ and

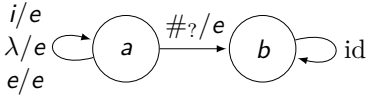
$u_i = v_i$

Remaining relations:

$u_0 \#_? u_1 \ldots \#_? u_k$

$=_\tau u_0 \#_? u_1 \ldots \#_? u_k$





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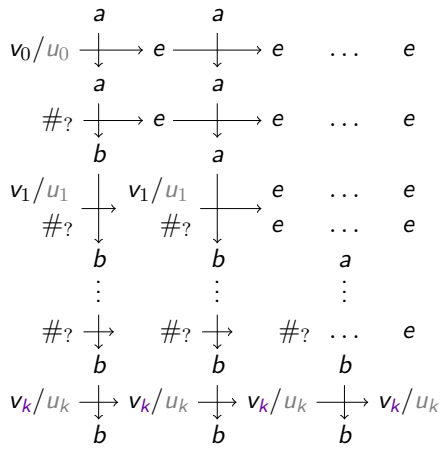


$\implies k = \ell$ and

$u_i = v_i$

Remaining relations:

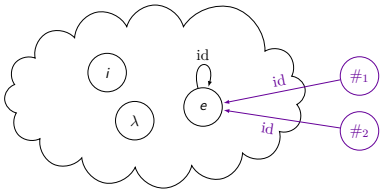
$u_0 \#_? u_1 \dots \#_? u_k$

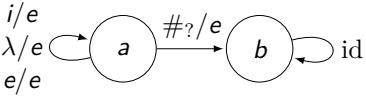
$=_{\mathcal{T}} u_0 \#_? u_1 \dots \#_? u_k$

Chop off blocks via a

$\rightsquigarrow \quad u \#_1 v \#_? x$

$=_{\mathcal{T}} u \#_2 v \#_? y$





(dual automaton)

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Reduction

Consider a relation

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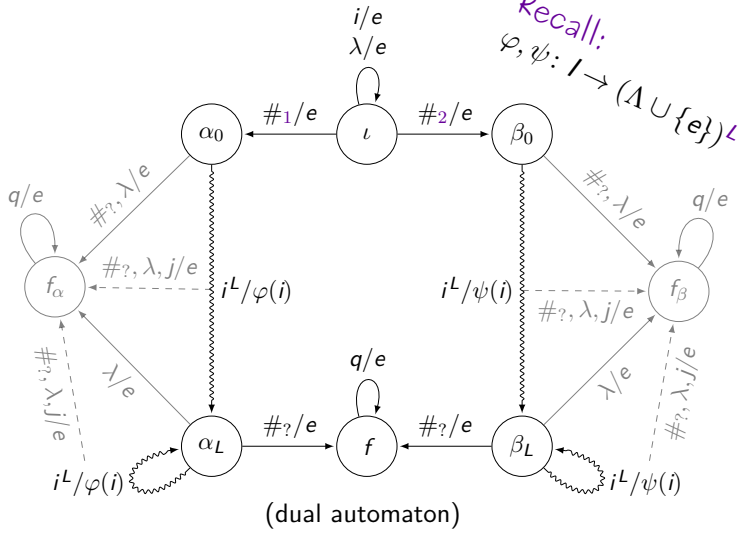

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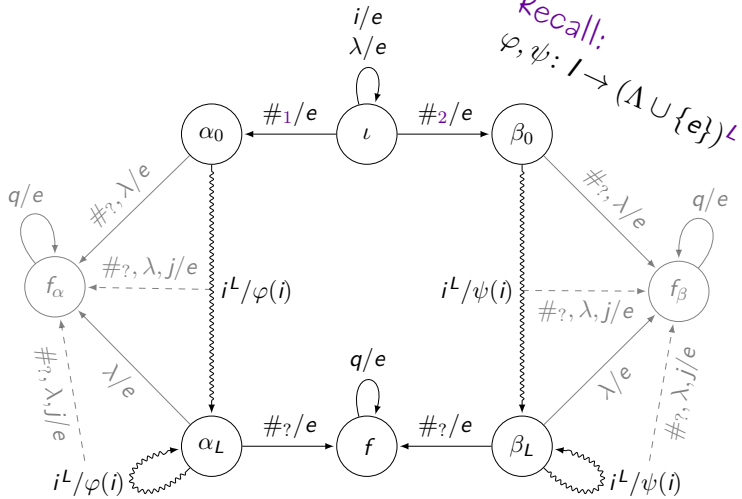
v	v
$\#_?$	$\#_?$
x	y



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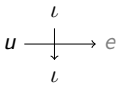

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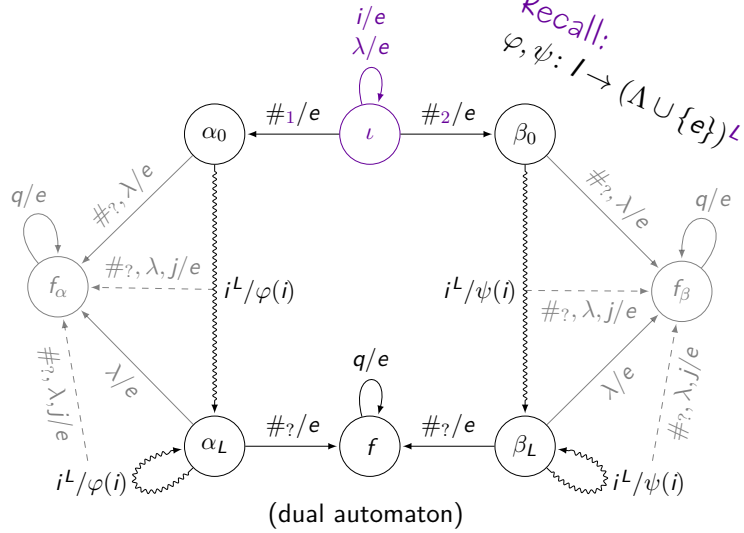

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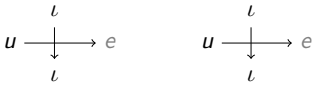

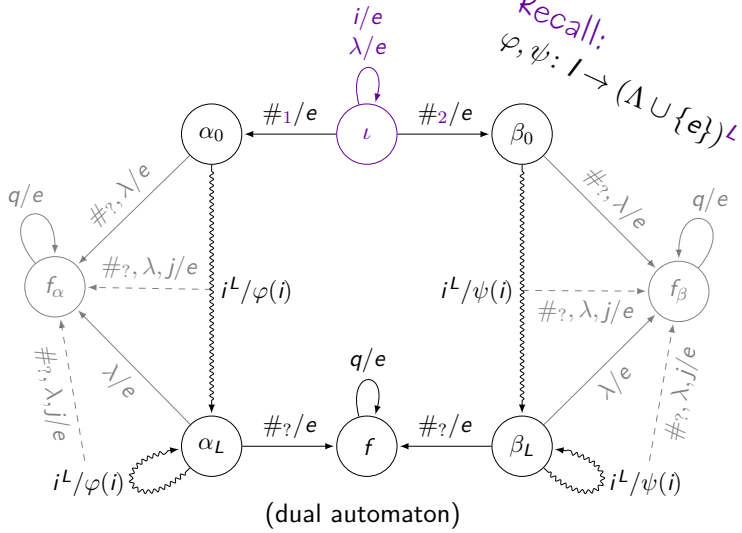

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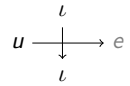
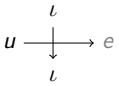
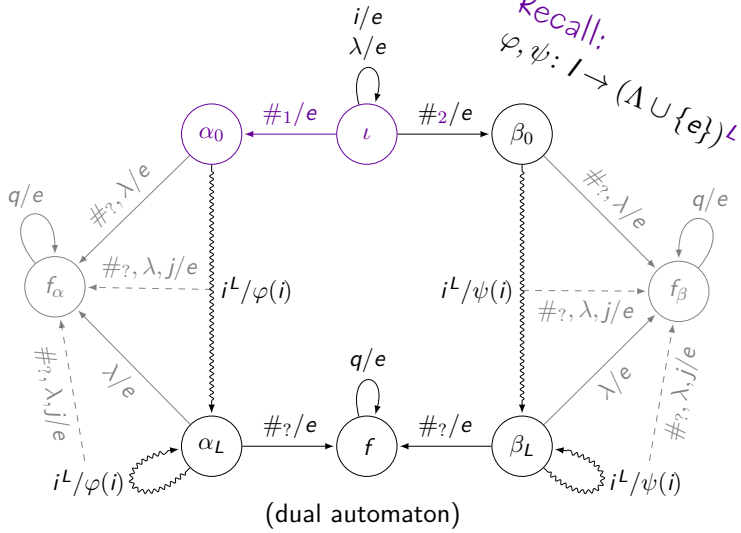




v $\quad$ v

$\#_?$ $\quad$ $\#_?$

x $\quad$ y



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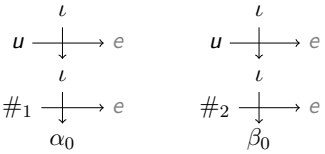

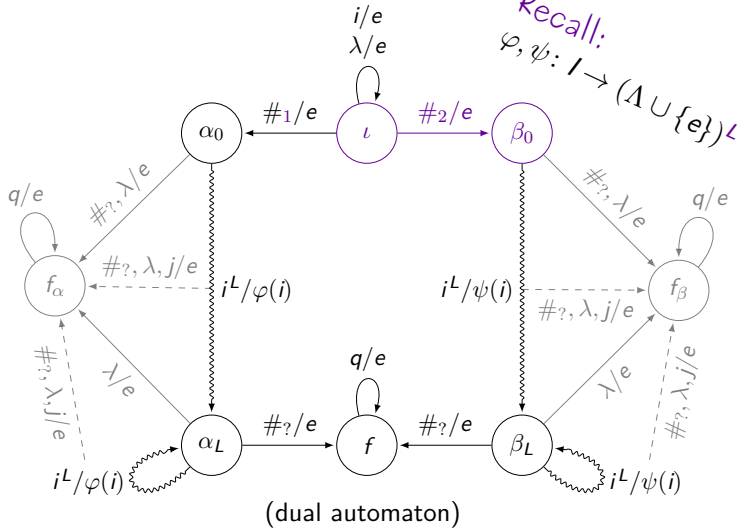

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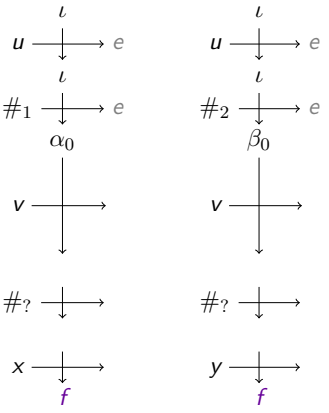

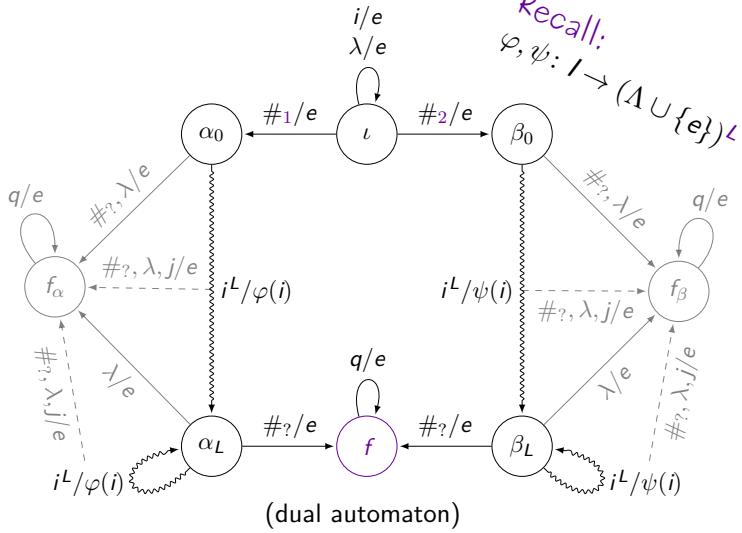

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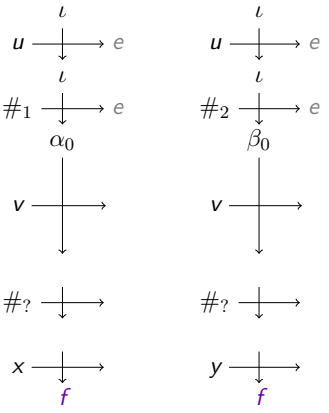

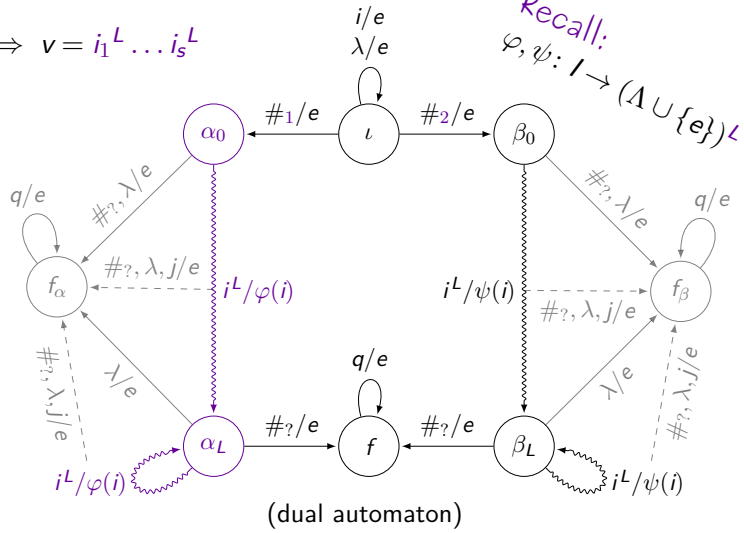

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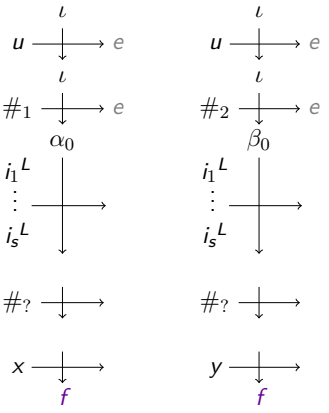

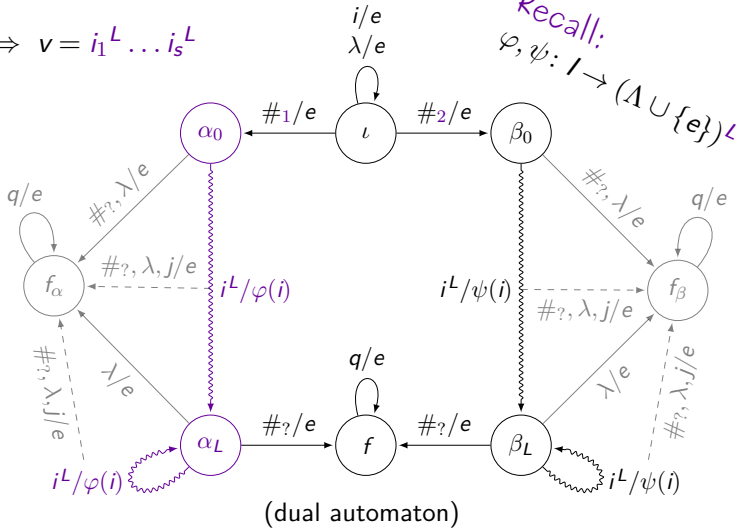

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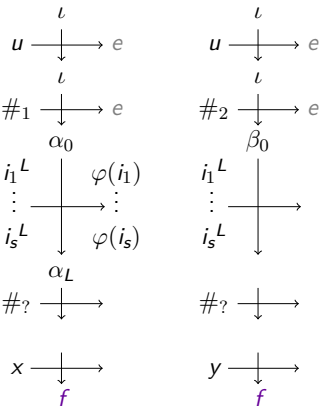

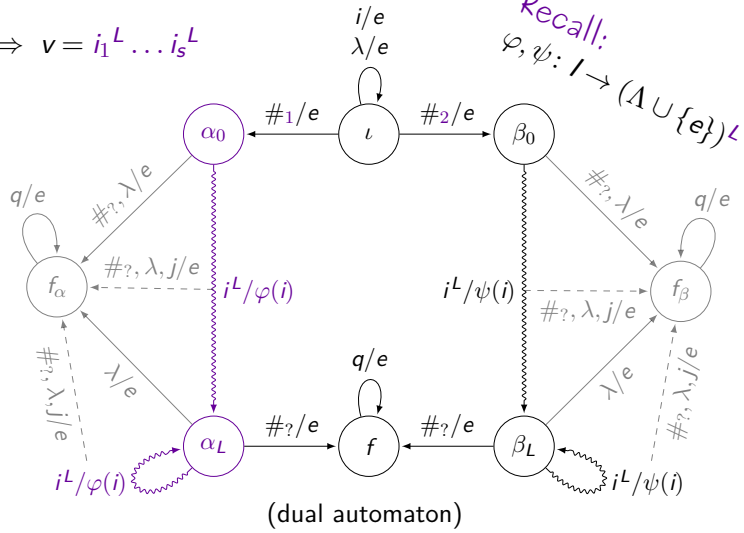

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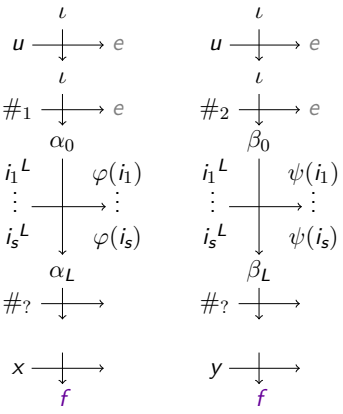

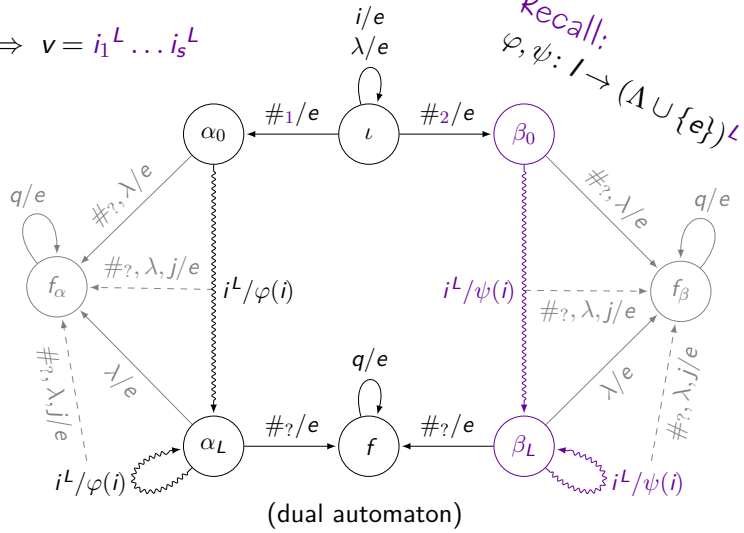

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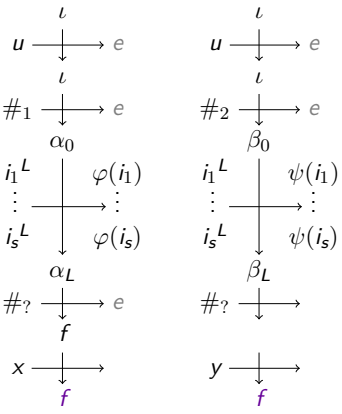

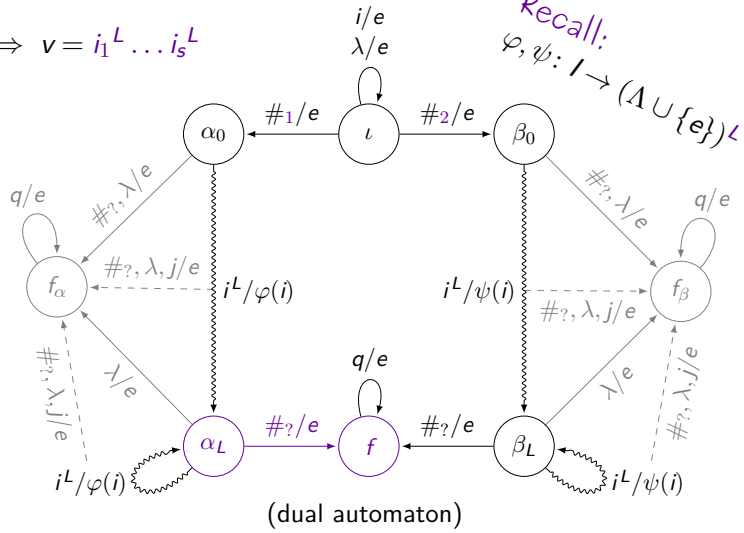

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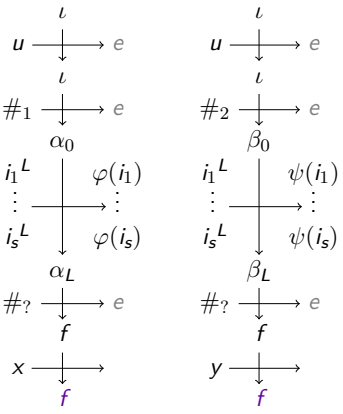

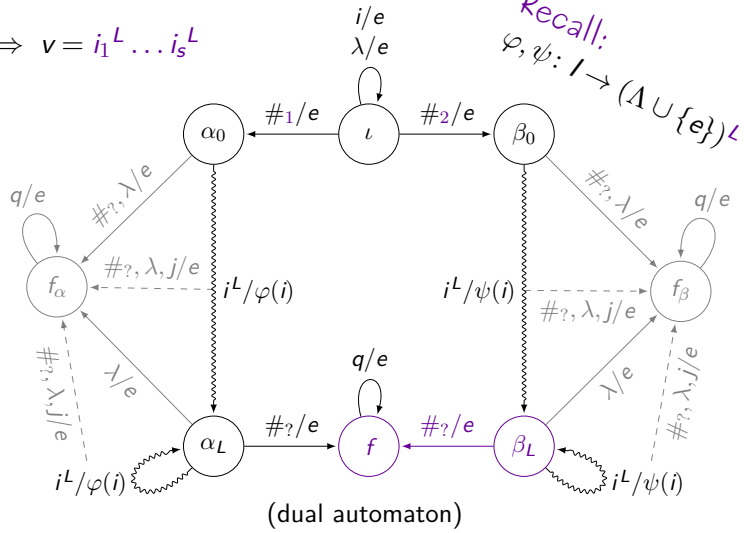

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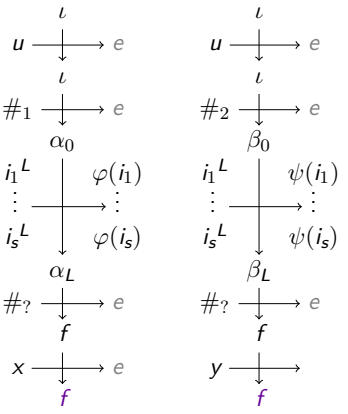

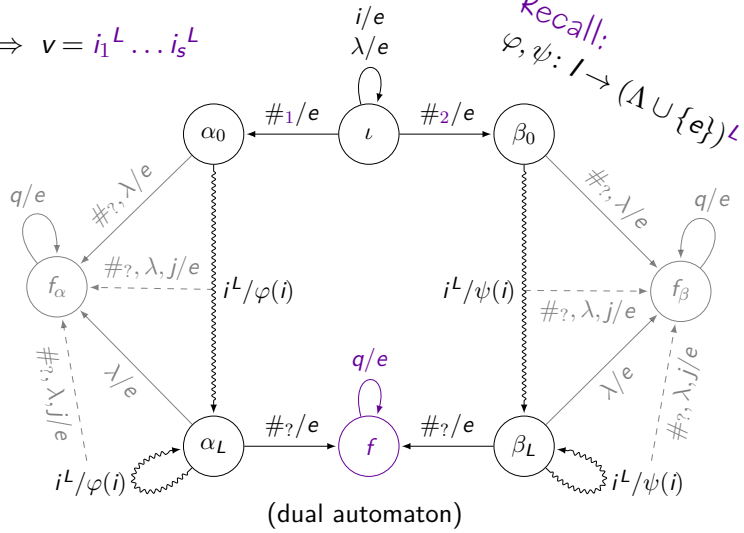

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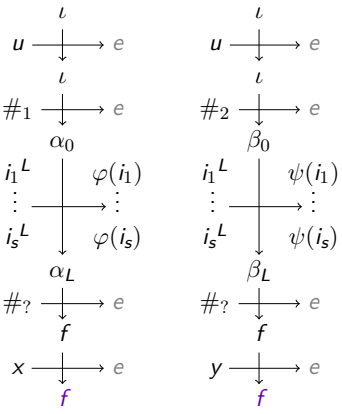

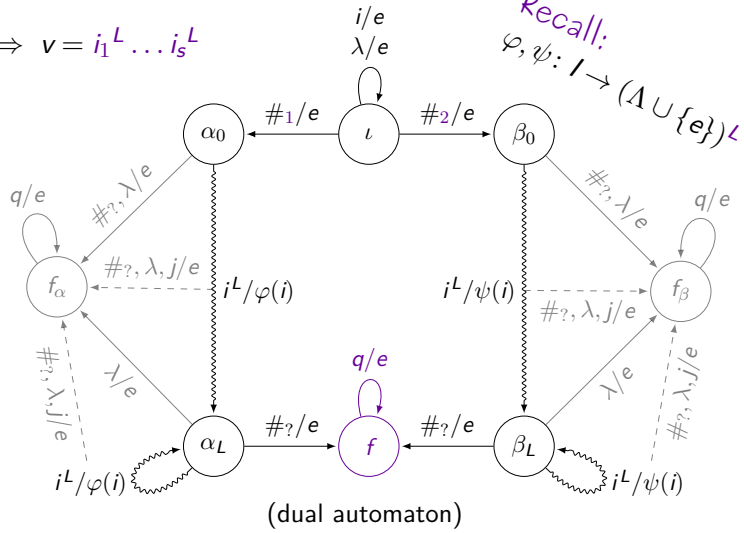

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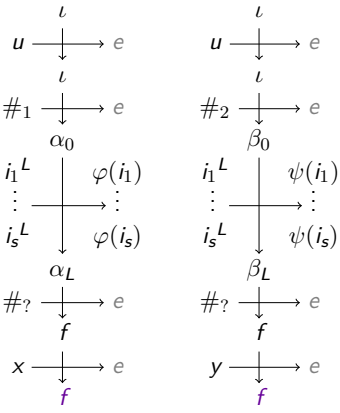

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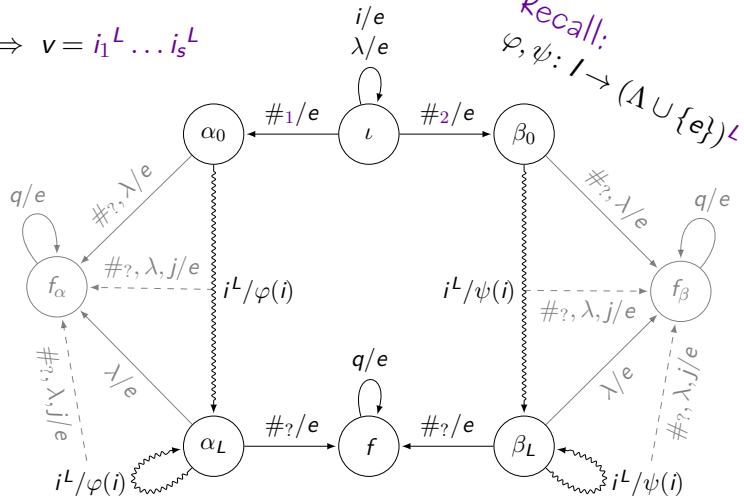

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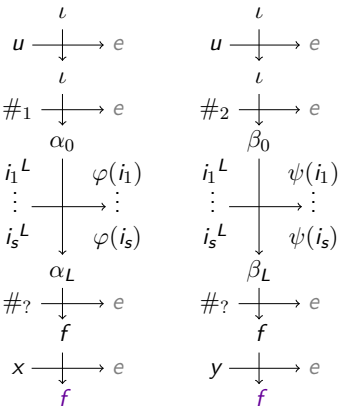

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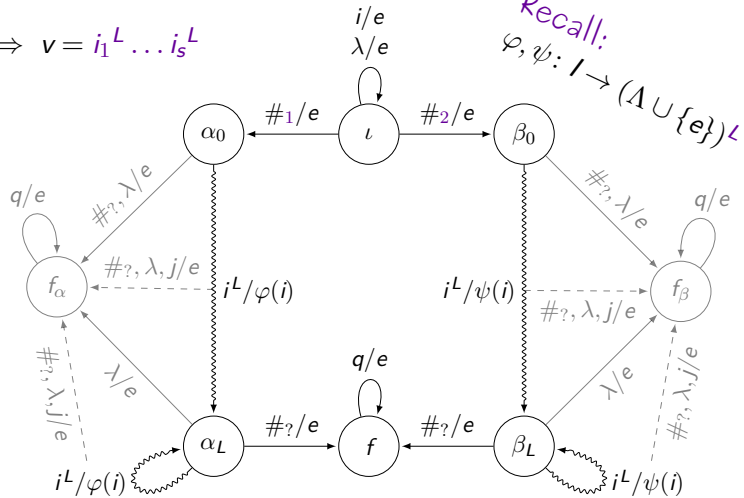

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Is there something similar for adding a free generator to groups?

Thank you!