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The Freeness Problem for Automaton Semigroups

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Algorithmic Problems in Group Theory

Max Dehn's three fundamental problems of algorithmic group theory:

Definition (Word Problem)

Constant: a group G
Input: a group element $g \in G$
Question: is $g = 1$?

Definition (Conjugacy Problem)

Constant: a group G
Input: two group elements $g, h \in G$
Question: $\exists k \in G : g = k^{-1} h k$ (i. e. are they conjugate)?

Definition (Isomorphism Problem)

Input: two groups G and H
Question: are G and H isomorphic?

The Freeness Problem

Today:

Definition (Freeness Problem)

Input: a group/monoid/semigroup
Question: is it a free group/monoid/semigroup?

i.e. $X \simeq F(B) \,/\, B^* \,/\, B^+$ for some basis B

But: How can we encode a (semi)group for an algorithm? Some options:

- traditional presentation: $\langle q_1, \dots, q_n \mid \ell_1 = r_1, \dots, \ell_m = r_m \rangle$
  $Q = \{q_1, \dots, q_n\}$: generators, $(\ell_1, r_1), \dots (\ell_m, r_m) \in Q^+ \times Q^+$: relations
  $\rightsquigarrow$ if both sets are finite: finitely presented (semi)group
- (invertible) matrices as generators
- We will use: automata $\rightsquigarrow$ automaton (semi)groups

Why use automata?

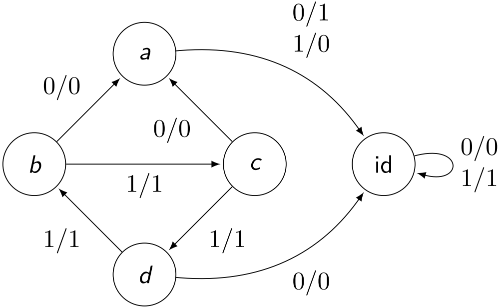
- Many examples of (semi)groups with interesting properties arise in this way and
- it allows for a finite description of possibly non-finitely presented (semi)groups.
  $\rightsquigarrow$ This makes them algorithmically interesting!

Let's look at a famous example!

Example: Grigorchuk's Group...

- ...is the historically first example of a group of intermediate growth (i. e. subexponential but superpolynomial).
  "Milnor Problem"
- ...is a Burnside group: all elements have finite order but the group is infinite.
  "Burnside Problem"
- ...is amenable but not elementary amenable. "Day Problem"
- ...is not finitely presented.

- But: It is generated by the automaton:



"How does this work?"
$\rightsquigarrow$ Later!

Automata and Algorithmic Questions

First: Let's look at what is known algorithmically!

Some known results:

Problem	Automaton Groups	Automaton Monoids/Semigroups
Word Problem	PSPACE-complete (W., Weiß 2020)	PSPACE-complete (D'Angeli, Rodaro, W. 2017)
Conjugacy Problem	undecidable (Šunić, Ventura 2012)	not applicable
Isomorphism Problem	undecidable (follows from Šunić, Ventura 2012)	
Finiteness Problem	*open*	undecidable (Gillibert 2014)
Freeness Problem	*open*	undecidable (D'Angeli, Rodaro, W. 2024)

These proofs are all somewhat ad-hoc…

Comparison: Traditional Presentations

For traditional finite presentations $\langle Q \mid \mathcal{R} \rangle$, these problems are all known to be undecidable.

Typically, this is done using Markov properties:

Note: we only talk about finitely presented groups here!

Definition (Markov property)

An abstract group property $\mathcal{P}$ is a Markov property if

- $\exists G_+ : G_+$ has $\mathcal{P}$ and
- $\exists G_- : G_- \hookrightarrow G \implies G$ does not have $\mathcal{P}$

Examples: finite, free, isomorphic to H

Theorem (Adian 1955; Rabin 1958)

For any Markov property $\mathcal{P}$, the problem

Constant:	*Markov property $\mathcal{P}$*
Input:	*a finite presentation $\langle Q \mid R \rangle$*
Question:	*has the presented group the property $\mathcal{P}$?*

is undecidable.

There is an earlier version for monoids.

Our Example: The Freeness Problem

Input: a group/monoid/semigroup
Question: is it a free group/monoid/semigroup?

Presentation	Freeness problem for… groups	monoids	semigroups
$\langle Q \mid \mathcal{R} \rangle$	undecidable (Markov property)	undecidable (Markov property)	decidable (eliminating relations)
matrices	*open*	undecidable (Klarner, Birget, Satterfield 1991)	
automata	*open*	undecidable (D'Angeli, Rodaro, W. 2024)	

There are many more (un)decidability results for special cases!

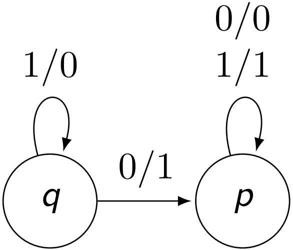
We will discuss the slightly simpler monoid case.

How can we use automata to generate (semi)groups?

Automata

In this setting, an automaton $\mathcal{T} = (Q, \Sigma, \delta)$ is a

- finite, directed graph whose
- nodes from Q are called states and
- edges given by $\delta \subseteq Q \times \Sigma \times \Sigma \times Q$ are called transitions and
- are labeled by pairs a/b of letters from the alphabet Σ.
- A transition $p \xrightarrow{a/b} q$
  - starts in p and
  - ends in q. Its
  - input is a and its
  - output is b.



An automaton is

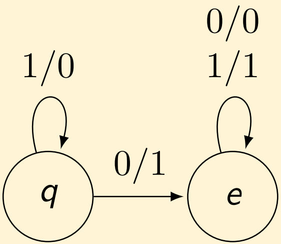
- deterministic if $\forall a \in \Sigma \, \forall q \in Q : q$ has at most one outgoing transition with input a.
- complete if $\forall a \in \Sigma \, \forall q \in Q : q$ has at least one outgoing transition with input a.

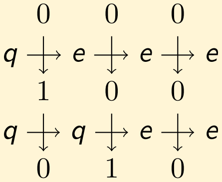
Today we only consider deterministic and complete automata!

State Actions

- Idea: every state sequence $\boldsymbol{q} \in Q^+$ induces an action $\Sigma^* \to \Sigma^*, u \mapsto \boldsymbol{q} \circ u$ mapping input words to output words.

Example



- The action of e is the identity.
- 

$$q \circ 000 = 100$$
$$qq \circ 000 = q \circ 100 = 010$$
$$qqq \circ 000 = \cdots = 110$$

$\rightsquigarrow$ the action of q increments (reverse) binary representation (least significant bit first)

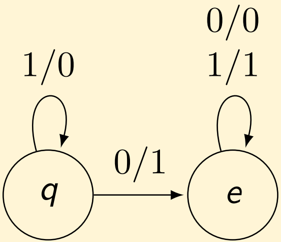
Automaton Semigroups, Monoids and Groups

- This defines a congruence

$$\boldsymbol{p} =_{\mathcal{T}} \boldsymbol{q} \iff \forall u \in \Sigma^* : \boldsymbol{p} \circ u = \boldsymbol{q} \circ u$$

- generated semigroup: $\mathscr{S}(\mathcal{T}) = Q^+ / {=_{\mathcal{T}}}$ "automaton semigroup"
- generated monoid: $\mathscr{M}(\mathcal{T}) = Q^* / {=_{\mathcal{T}}}$ "automaton monoid"
- For groups: add inverses

Example



- e: identity
- q: increment
- $qe =_{\mathcal{T}} eq =_{\mathcal{T}} q$
- $q^i \neq_{\mathcal{T}} q^j$ for $i \neq j$

$$\mathscr{M}(\mathcal{T}) = \mathscr{S}(\mathcal{T}) \simeq q^*$$
$$\mathscr{G}(\mathcal{T}) = F(q) \simeq \mathbb{Z}$$

Relations and Cross Diagrams

We have: $\boldsymbol{q} = q_1 \dots q_\ell =_{\mathcal{T}} p_1 \dots p_k = \boldsymbol{p}$ (i. e. a relation) $\iff$

$$
\forall w: \quad
\begin{array}{ccccc}
 & w & & w' & \\
q_1 & \longrightarrow & q_1' & \longrightarrow & q_1'' \\
 & u_1 & & u_1' & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
 & u_{\ell-1} & & u_{\ell-1}' & \\
q_\ell & \longrightarrow & q_\ell' & \longrightarrow & q_\ell'' \\
 & u & & u' &
\end{array}
\qquad
\begin{array}{ccccc}
 & w & & w' & \\
p_1 & \longrightarrow & p_1' & \longrightarrow & p_1'' \\
 & v_1 & & v_1' & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
 & v_{k-1} & & v_{k-1}' & \\
p_k & \longrightarrow & p_k' & \longrightarrow & p_k'' \\
 & v & & v' &
\end{array}
\implies u = v
$$

$\implies \boldsymbol{q}' = q_1' \dots q_\ell' =_{\mathcal{T}} p_1' \dots p_k' = \boldsymbol{p}'$ is a relation

Dual Automaton

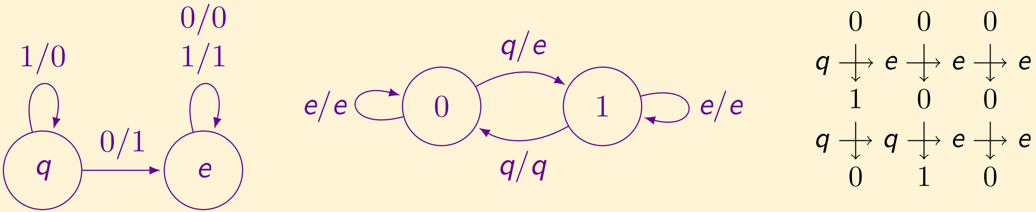
Idea: Swap letters and states

Definition

$\partial\mathcal{T} = (\Sigma, Q, \partial\delta)$ with

$$\partial\delta = \{a \xrightarrow{p/q} b \mid p \xrightarrow{a/b} q \in \delta\}$$

Example



The Freeness Problem

Main Theorem

Theorem

The following problem is undecidable for given automaton semigroups and monoids:

Input: an $\mathscr{S}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$

Question: is $\mathscr{S}(\mathcal{T}) / \mathscr{M}(\mathcal{T})$ free?

$st = st' \implies t = t'$

(left) cancellative? equidivisible?

$\mathscr{M}(\mathcal{T}) \simeq (Q \setminus \{\mathrm{id}\})^*$?

- The proof is based on a reduction from Post's Correspondence Problem and
- yields further results.

Lemma (Levi's Lemma)

A semigroup (monoid) if free if and only if

1. *it has a (proper) length function and*
2. *is equidivisible.*

What about this part? ⇝ WIP

Reductions

Definition (Decision Problem)

Decision problem A:

Input: $x_1, \ldots, x_n$ ← This is called an instance of A

Question: $(x_1, \ldots, x_n) \in A$?

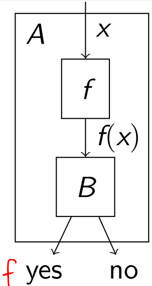
There are positive ($\in A$) and negative ($\notin A$) instances

Definition (many-one reduction)

$A \leq B \iff$ there is a computable total function f mapping instances of A to instances of B such that

“A reduces to B”

$$A \ni x \iff f(x) \in B.$$



Idea: An algorithm for B also yields one for A.

In particular: If A is undecidable, so is B. *the complement of

Thus: Reduce an undecidable problem to* our freeness problem…

Post's Correspondence Problem

Definition (Post's Correspondence Problem)

Constant: an alphabet Λ and a padding symbol $e \notin \Lambda$

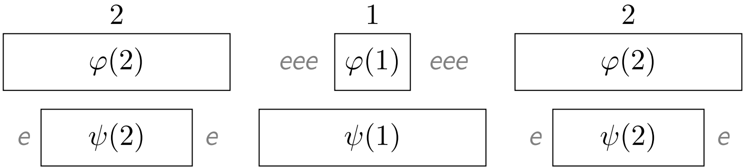
Input: a number L and
maps $\varphi, \psi \colon I = \{1, \ldots, n\} \to (\Lambda \cup \{e\})^L$

"all entries have length L"

Question: $\exists \boldsymbol{i} \in I^+ : \varphi(\boldsymbol{i}) =_e \psi(\boldsymbol{i})$? (extended into hom.)

Idea: We have $n = |I|$ many tiles and must "match" the upper with the lower entries.

Example:



PCP is a classical undecidable problem!

Proof

The Reduction

Definition (Post's Correspondence Problem)

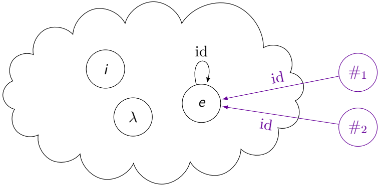
Constant: an alphabet Λ and a padding symbol $e \notin \Lambda$

Input: a number L and
maps $\varphi, \psi \colon I = \{1, \ldots, n\} \to (\Lambda \cup \{e\})^L$

Question: $\exists \boldsymbol{i} \in I^+ : \varphi(\boldsymbol{i}) =_e \psi(\boldsymbol{i})$? (extended into hom.)

Reduction:

- start with an automaton for $(\Lambda \cup I)^*$ with identity state e
- add two new identity states $\#_1$ and $\#_2$

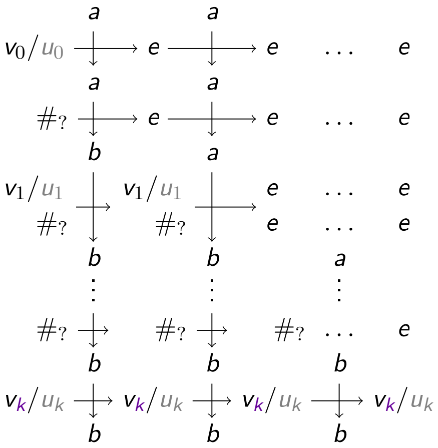


Relations: $u_0 \#_? u_1 \ldots \#_? u_k =_{\mathcal{T}} v_0 \#_? v_1 \ldots \#_? v_\ell$

$\iff u_0 \ldots u_k = v_0 \ldots v_\ell$

Now: ensure equality in the blocks...

Blocks in a Relation



$\Longrightarrow$ $k = \ell$ and

$u_i = v_i$

Remaining relations:

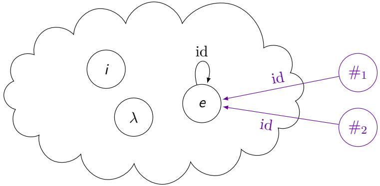
$u_0 \#_? u_1 \ldots \#_? u_k$

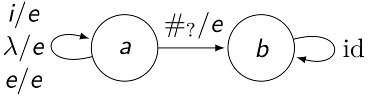
$=_\mathcal{T} u_0 \#_? u_1 \ldots \#_? u_k$

Chop off blocks via a

$\rightsquigarrow$ $u \#_1 v \#_? x$

$=_\mathcal{T} u \#_2 v \#_? y$





(dual automaton)

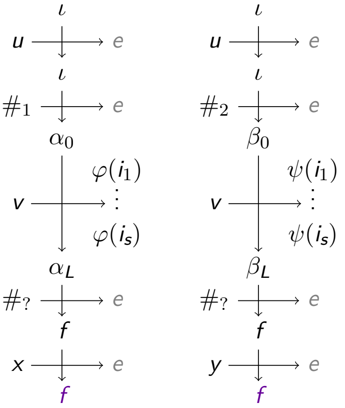
Why? $\rightsquigarrow$ It is a bit simpler & easier to understand.

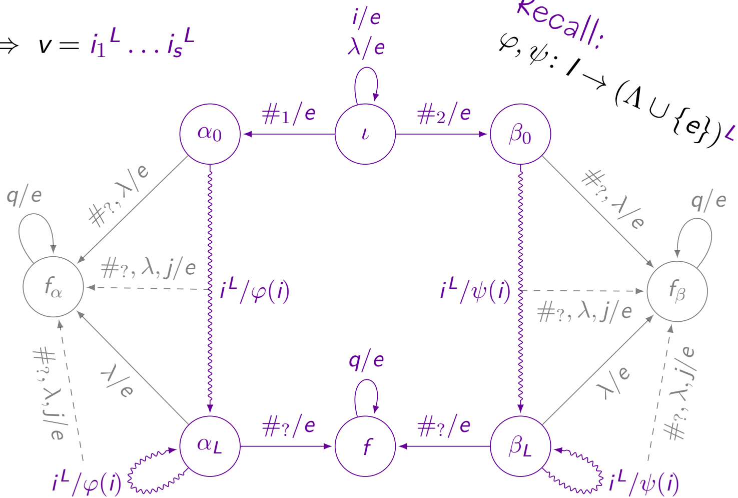
Reduction

Consider a relation

$$u \#_1 v \#_? x =_{\mathcal{T}} u \#_2 v \#_? y \implies v = {i_1}^L \dots {i_s}^L$$

Recall: $\varphi, \psi \colon I \to (\Lambda \cup \{e\})^L$





(dual automaton)

$$\implies \varphi(i_1 \dots i_s) =_e \psi(i_1 \dots i_s)$$

Relation and Solution

We have shown:

Proposition

$$\begin{aligned} & \mathscr{M}(\mathcal{T}) \text{ is \textbf{not} free} \\ \implies & \text{there is a proper relation} \\ \implies & u\#_1 i_1{}^L \dots i_s{}^L \#_1 =_{\mathcal{T}} u\#_2 i_1{}^L \dots i_s{}^L \#_1 \\ \implies & i_1 \dots i_s \text{ is a PCP solution} \end{aligned}$$

Proposition

$$\begin{aligned} & i_1 \dots i_s \text{ is a PCP solution} \\ \implies & \#_1 i_1{}^L \dots i_s{}^L \#_1 =_{\mathcal{T}} \#_2 i_1{}^L \dots i_s{}^L \#_1 \\ \implies & \mathscr{M}(\mathcal{T}) \text{ is \textbf{not} cancellative} \\ \implies & \mathscr{M}(\mathcal{T}) \text{ is \textbf{not} free} \end{aligned}$$

Future Work and Open Problems

- What about the free presentation problem for semigroups?
- What about having a length function?
- At what activity level does the problem become undecidable?
  Decidable for bounded activity monoids?
- We use these constructions:

Theorem (Macallister Brough, W., Welker 2025)

If S and T are automaton semigroups with a homomorphism $S \to T$ or $T \to S$, then the free product $S \star T$ is an automaton semigroup.

Corollary (of the construction)

If S is an automaton semigroup, also $S \star q^{+}$ is. The construction is effective.

Is there something similar for adding a free generator to groups?

Thank you!