Maximal Subgroups of Special Inverse Monoids II

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joint work with
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1 July 2025

Special Monoids and the Inverse E-Unitary Case

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Theorem (Malheiro; 2005)

M: special (ordinary, non-inverse) monoid
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What about arbitrary (non-E-unitary) inverse monoids?

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Theorem (Ivanov, Margolis, Meakin; 2001)

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2. the automorphism group of $S\Gamma(e)$ is G.

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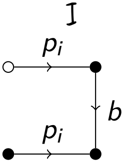
Graphically:



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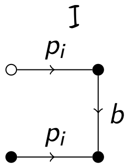

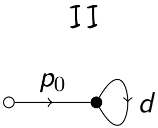

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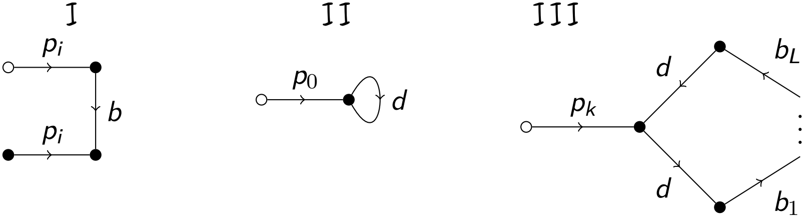
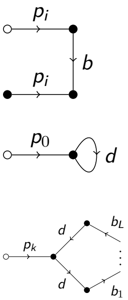


where $r_k = b_1 \dots b_L$

Relations:



Idea of the Construction

- Consider the idempotent $e = p_0^{-1} p_0 \prod_{k=1}^{R} p_k^{-1} p_k$:

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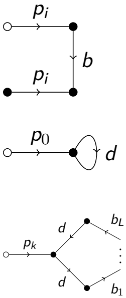

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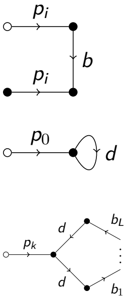
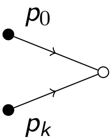


- 

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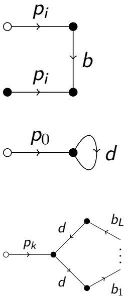
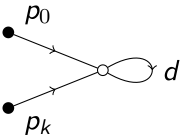


- 

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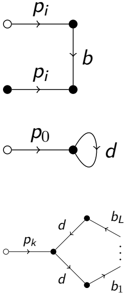
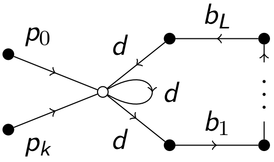


- 

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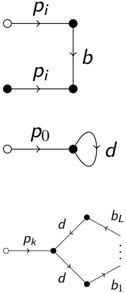
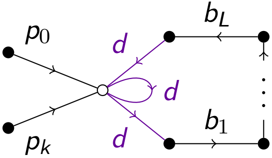


- 

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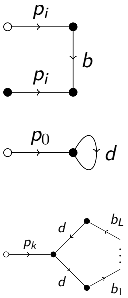
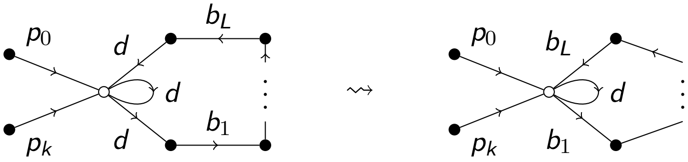


- 

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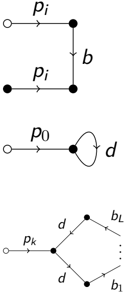
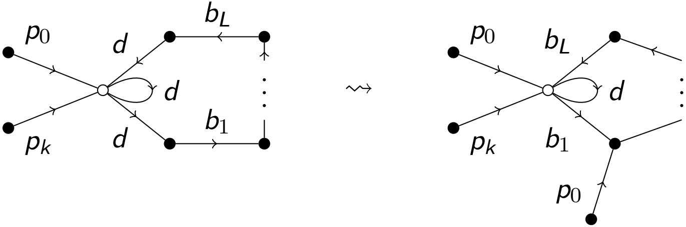


- 

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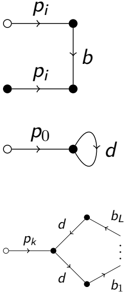
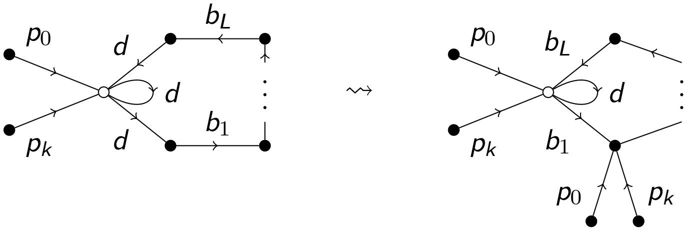


- 

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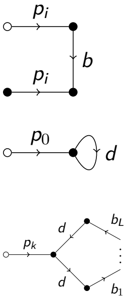
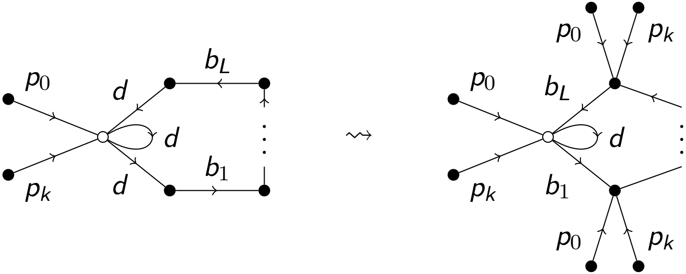


- 

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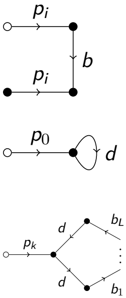
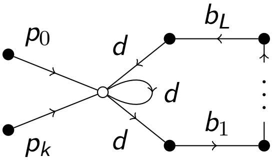
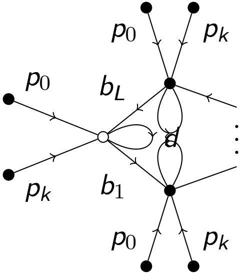


- 

$\rightsquigarrow$



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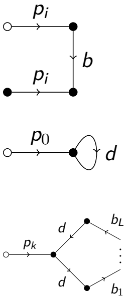
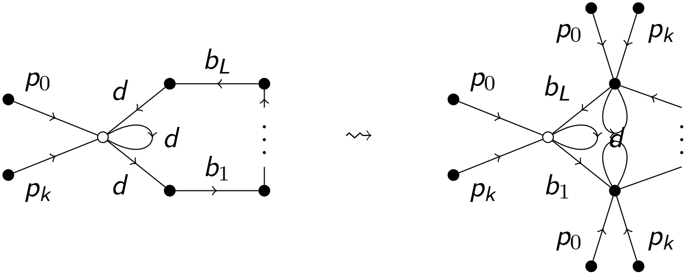


- 

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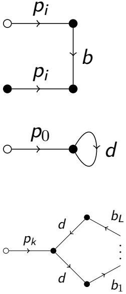
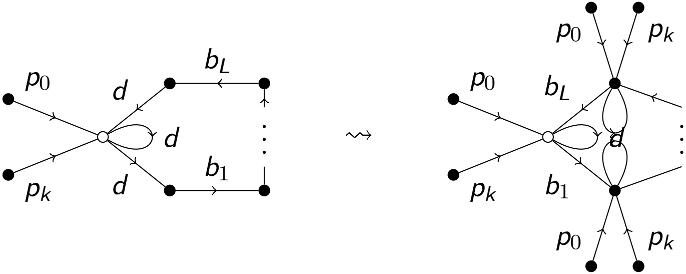


- 

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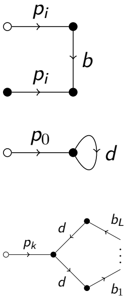
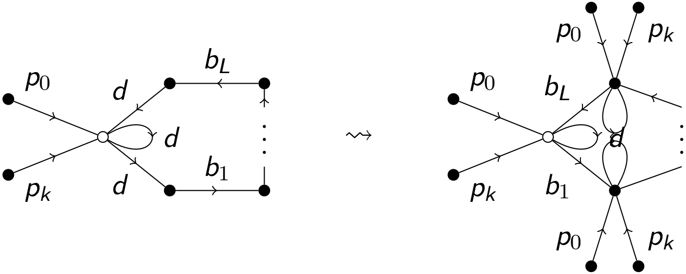


- 

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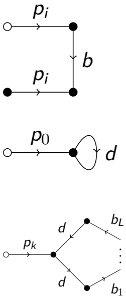
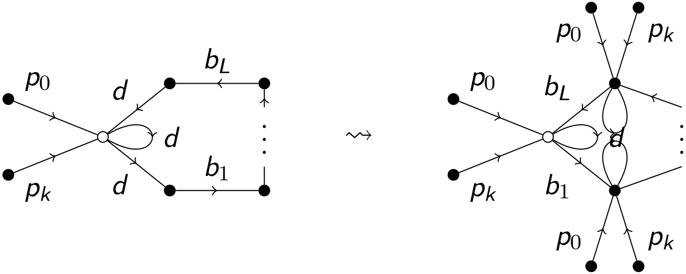


- 

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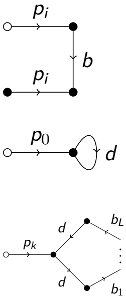
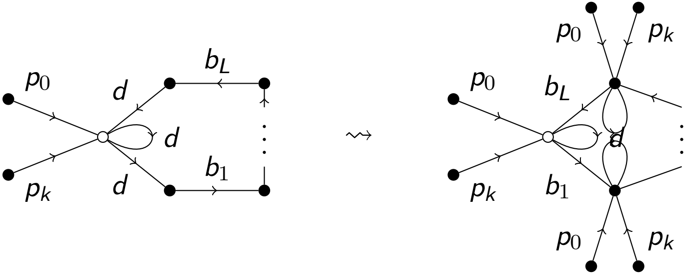


- 

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- How can we make this formal? We need an appropriate description!

A Grammar to Describe Tree-Like Graphs

Example

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$V = \{A, B\}$: set of nonterminals

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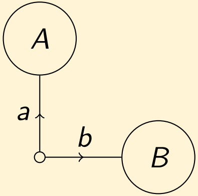
$A \to$ 

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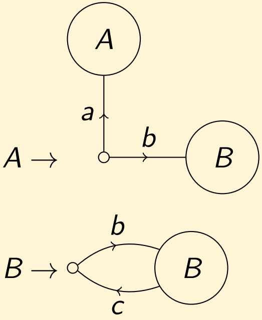

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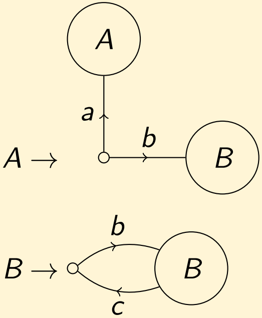

Generated Graph:

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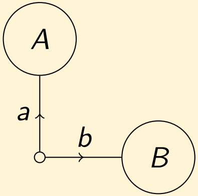
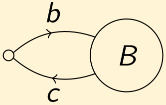
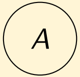
$V = \{A, B\}$: set of nonterminals $\Sigma = \{a, b\}$: edge labels

Rules: Generated Graph:

$A \to$ 

$B \to$ 



Example

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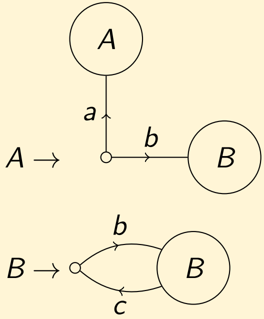

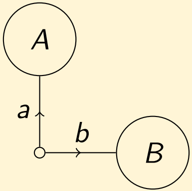

A Grammar to Describe Tree-Like Graphs

Example

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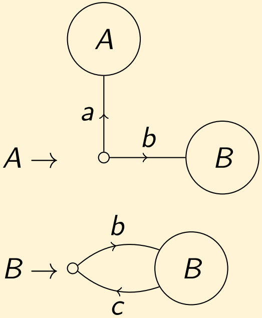
Rules:



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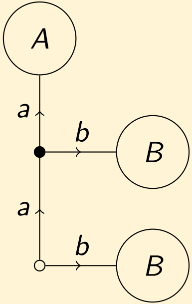

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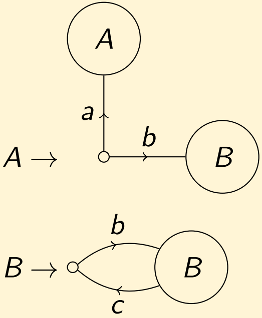

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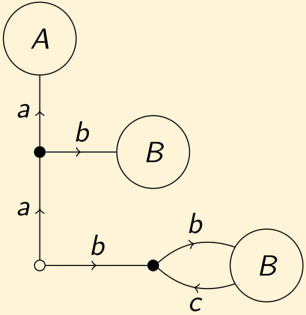

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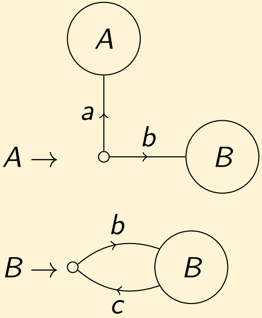

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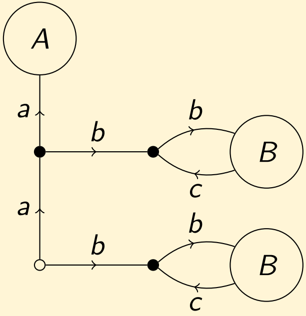

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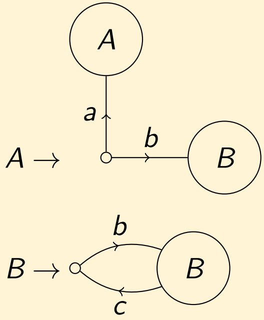
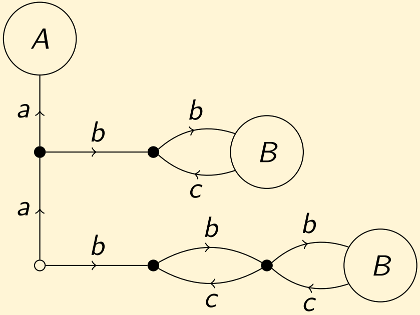


Generated Graph:



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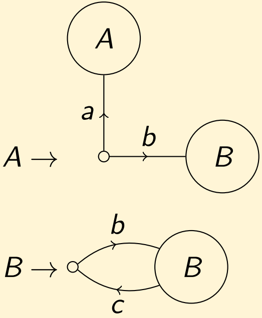
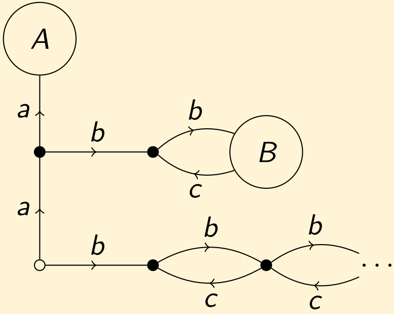


Generated Graph:



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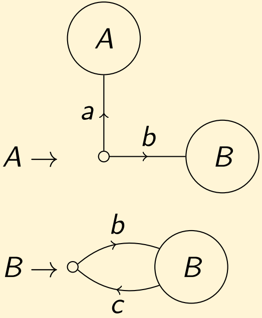
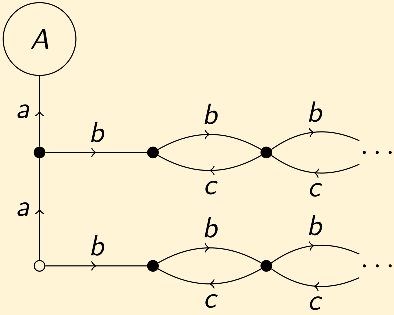


Generated Graph:



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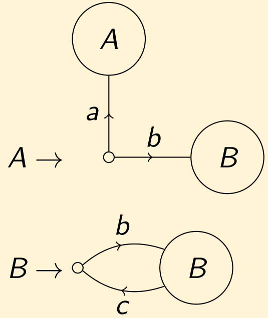
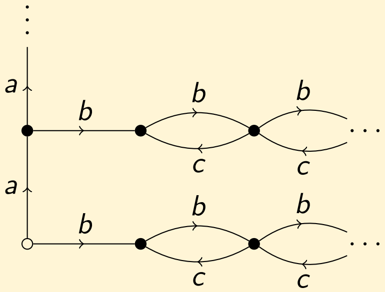
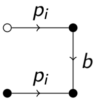


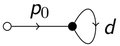
Generated Graph: vs "intermediate graphs"

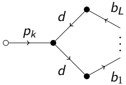


A Grammar for $S\Gamma(1)$

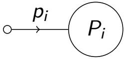
Relations:

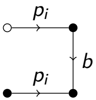


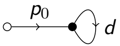


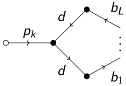


A Grammar for $S\Gamma(1)$

Relations: $1 \rightarrow$ 







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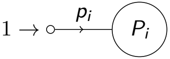
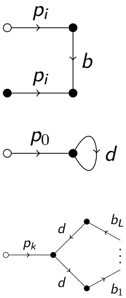
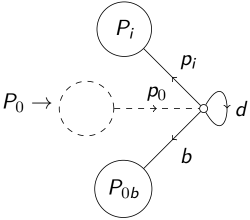
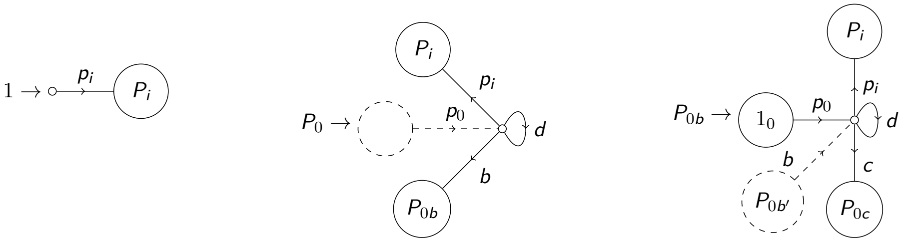


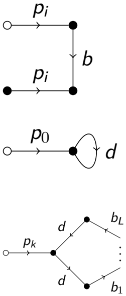




A Grammar for $\mathcal{ST}(1)$



Relations:



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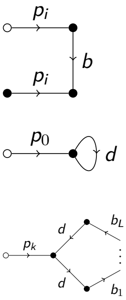

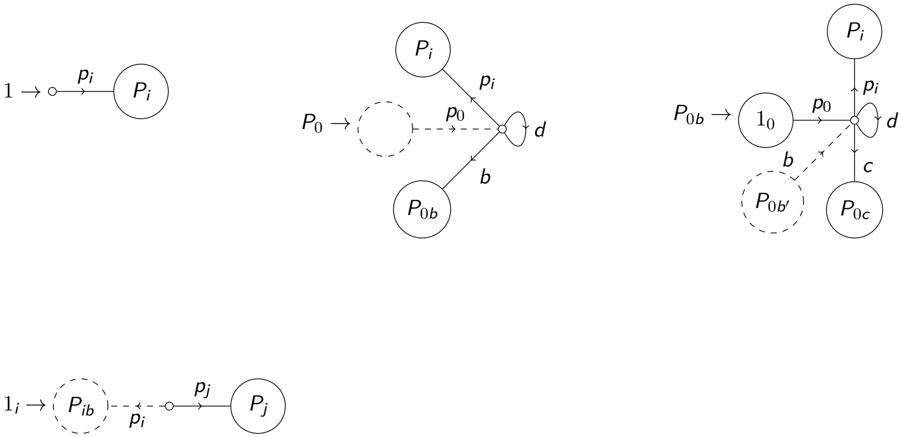

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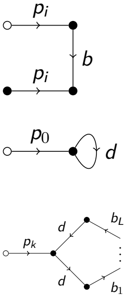

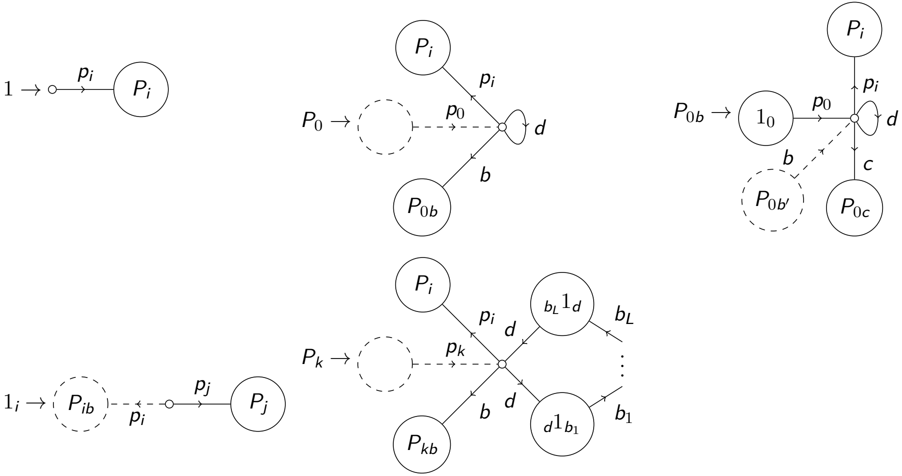

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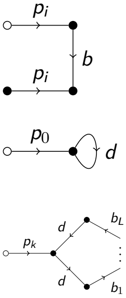
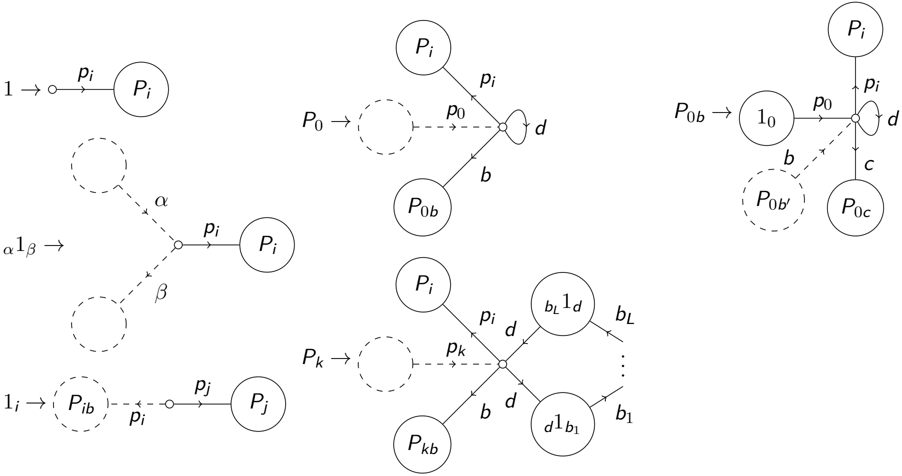

p_i
b
p_i
p_0
d
p_k
d
b_L
d
b_1


$1 \to$
p_i
P_i
${}_\alpha 1_\beta \to$
α
β
p_i
P_i
$1_i \to$
P_{ib}
p_i
p_j
P_j
$P_0 \to$
P_i
p_i
p_0
d
b
P_{0b}
$P_k \to$
P_i
p_i
p_k
d
${}_{b_L}1_d$
b_L
b
d
${}_d1_{b_1}$
b_1
P_{kb}
$P_{0b} \to$
1_0
p_0
P_i
p_i
d
b
c
$P_{0b'}$
P_{0c}

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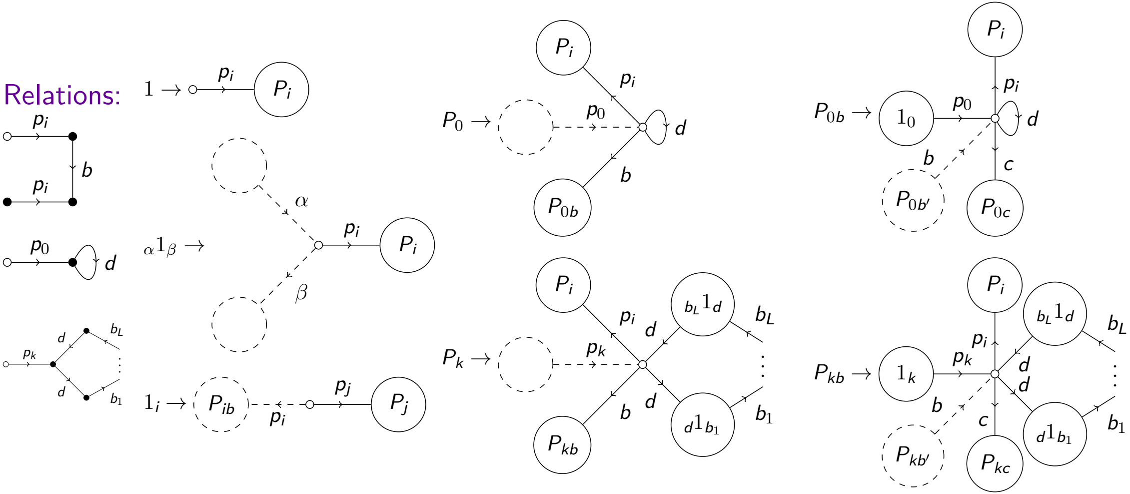

How do we Know that this Indeed Generates $S\Gamma(1)$?

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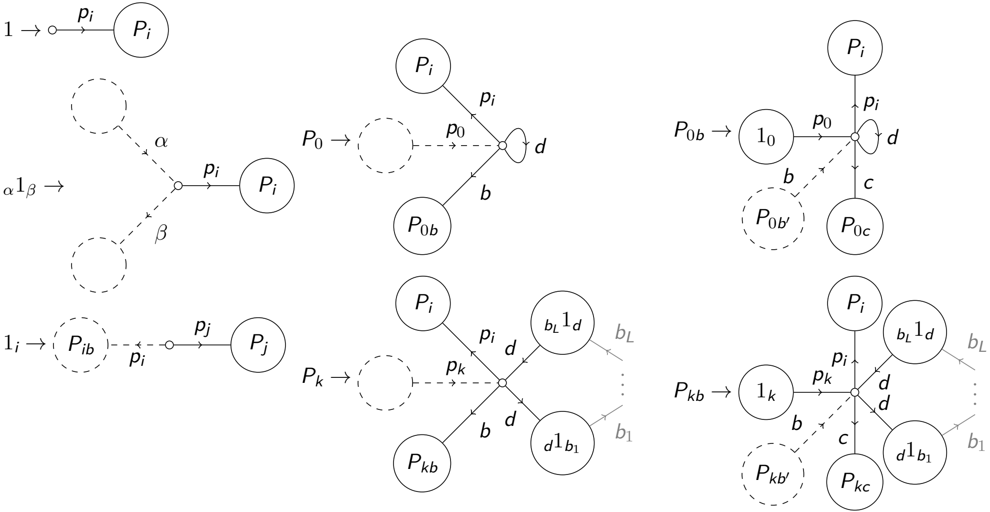
Note: We only have to show that the root must be mapped to itself.

Again: The Grammar for $S\Gamma(1)$



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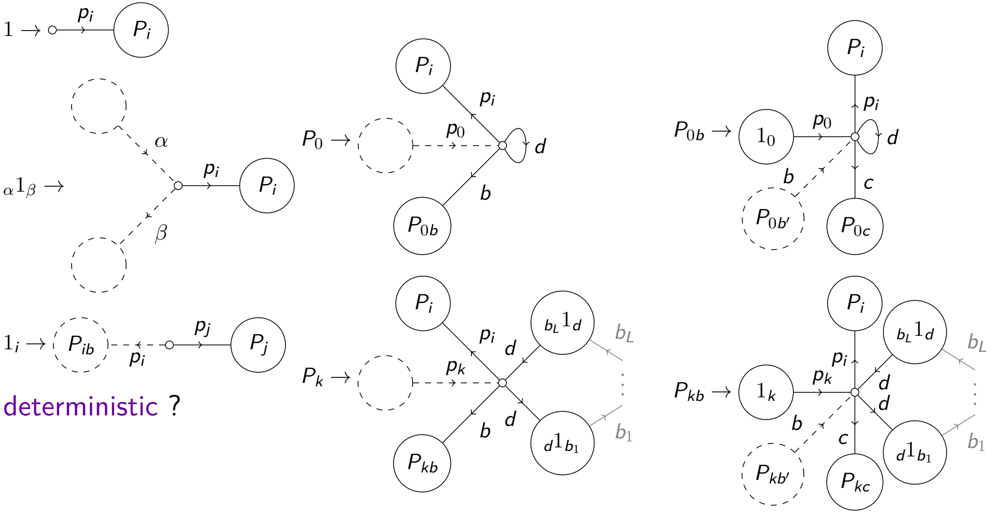
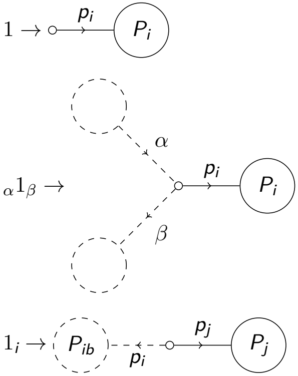
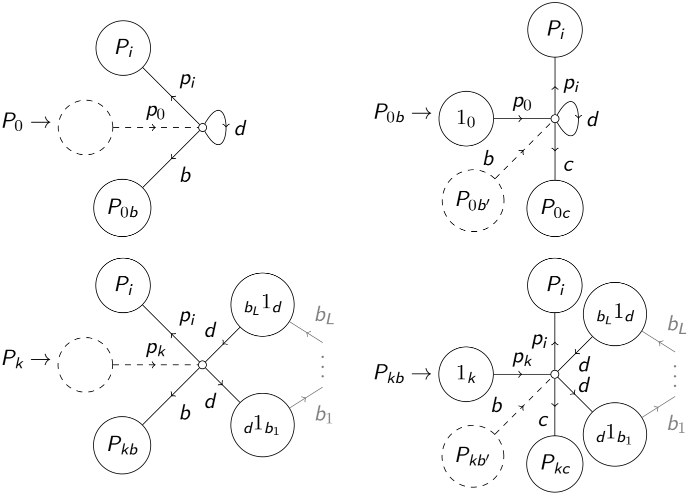


deterministic ?

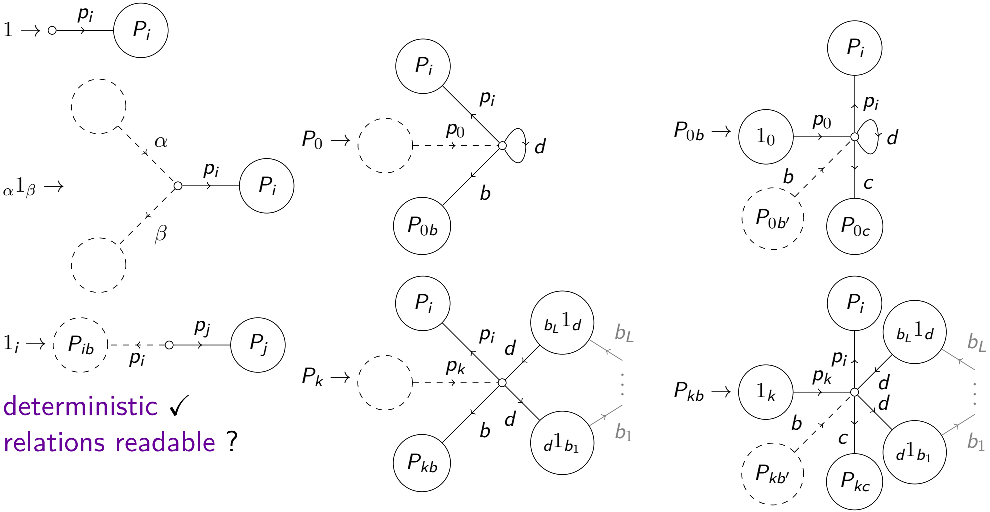
Again: The Grammar for $\mathcal{ST}(1)$



deterministic ✓



Again: The Grammar for $\mathcal{ST}(1)$



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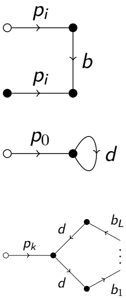
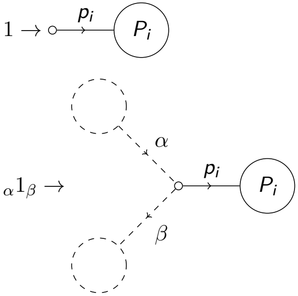
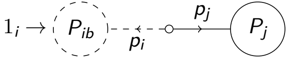
relations readable ?

Relations:







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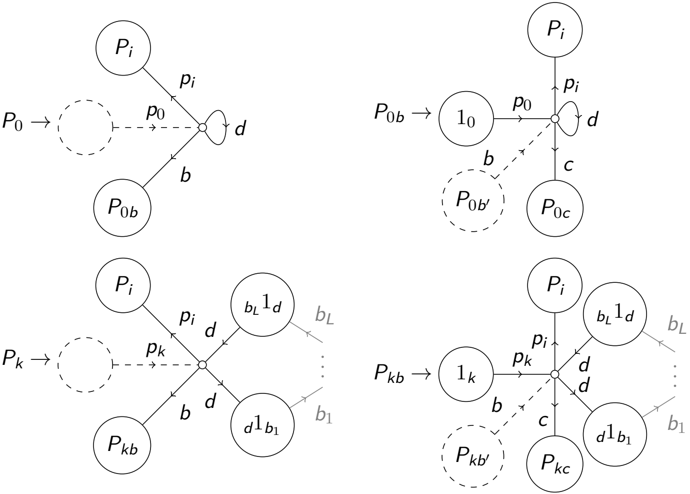

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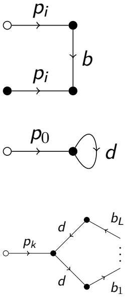

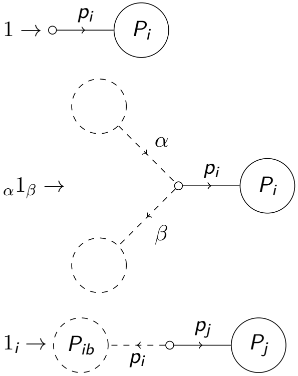

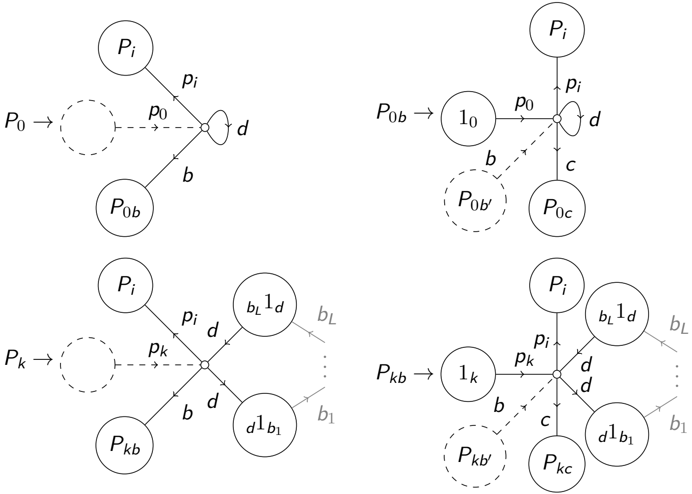

deterministic ✓

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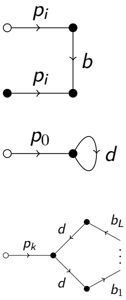

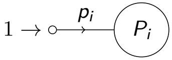

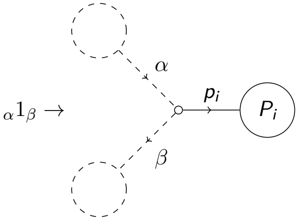

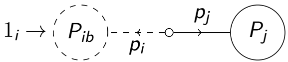

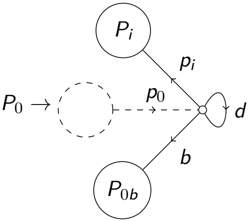

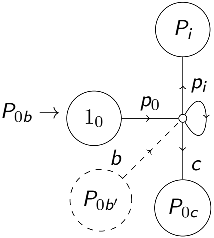

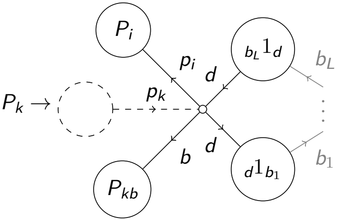

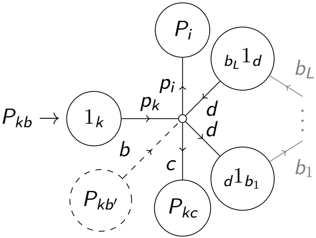

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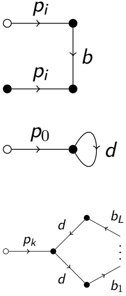

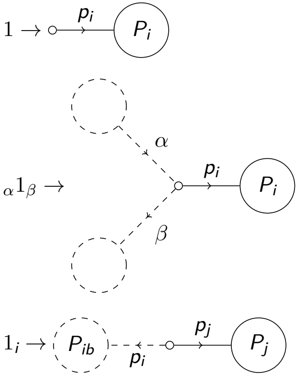

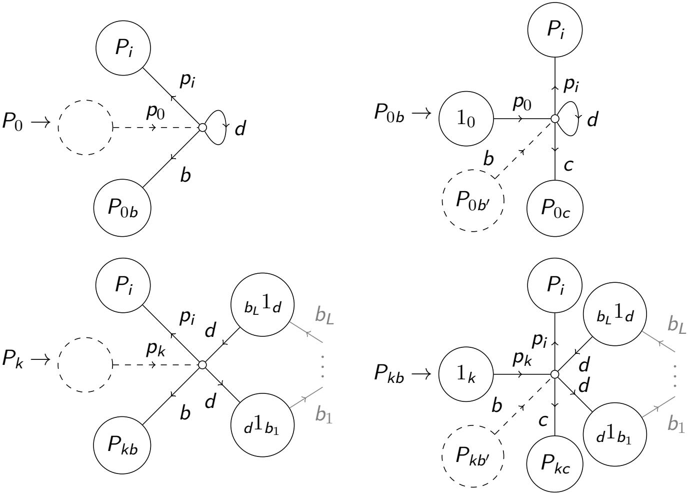

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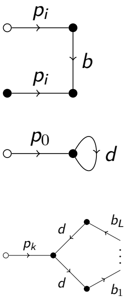

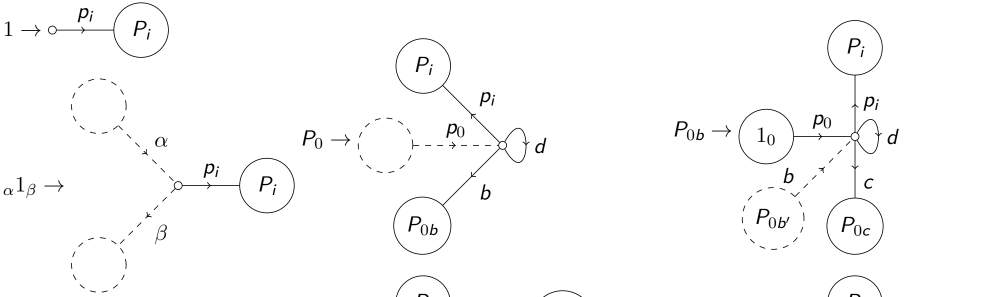

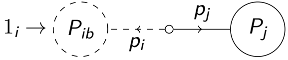

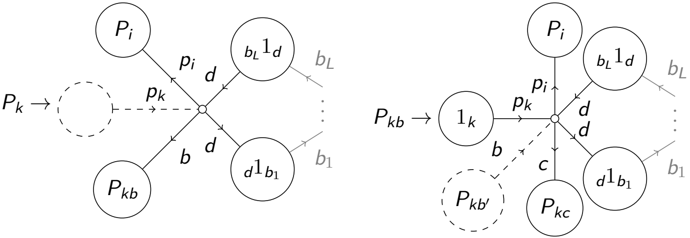

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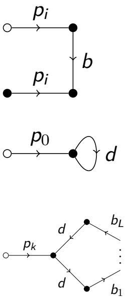

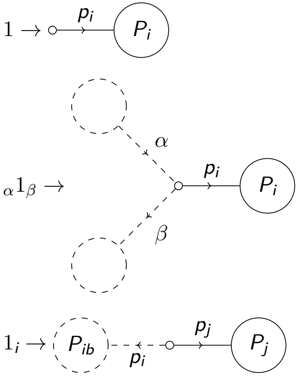

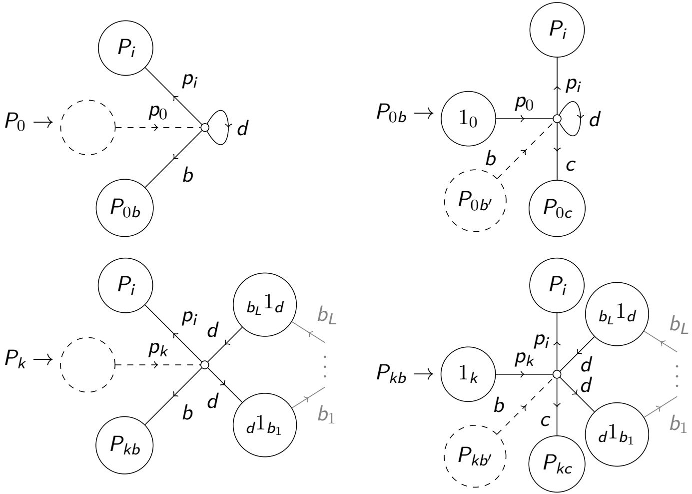

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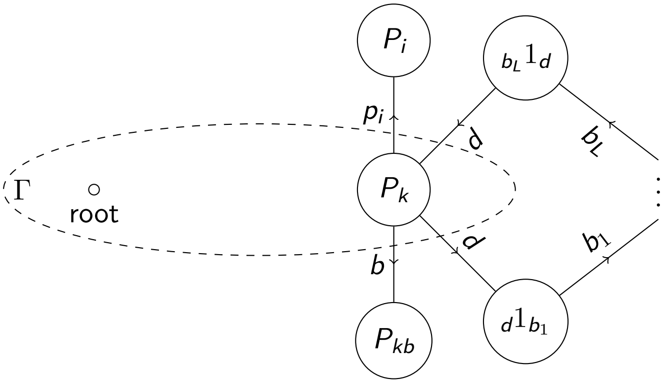

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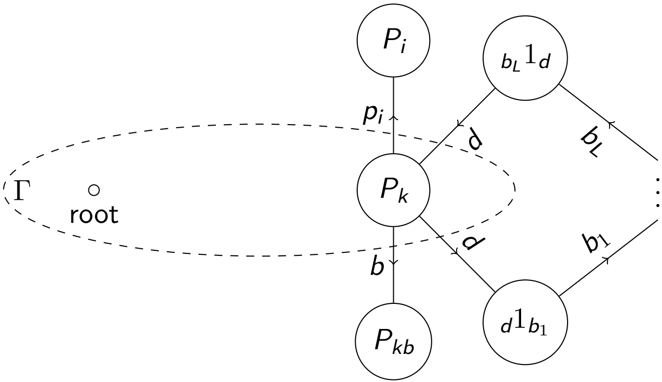

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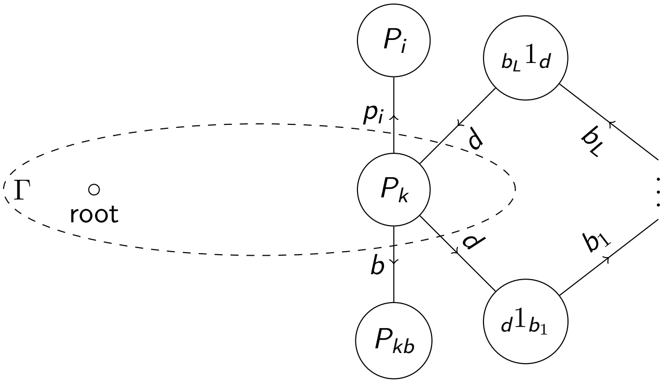

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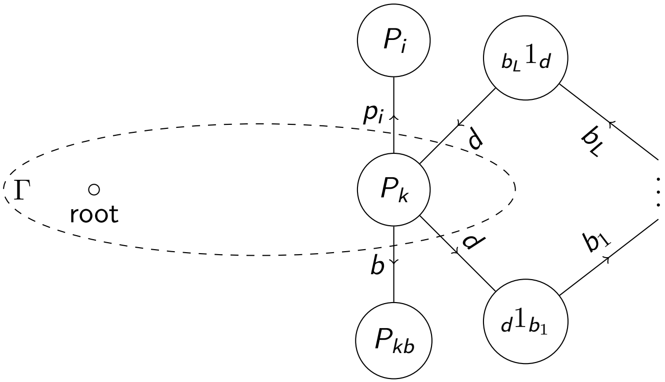

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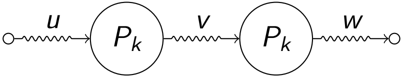

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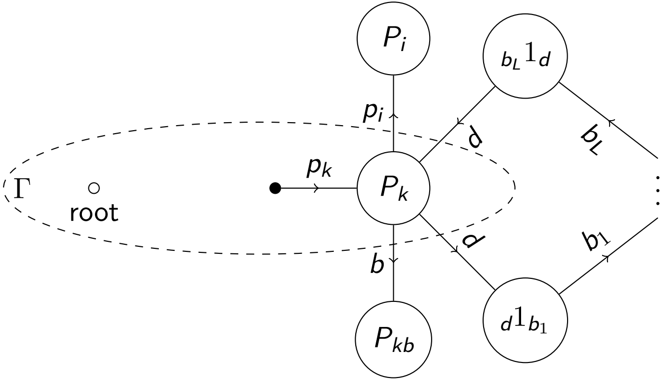

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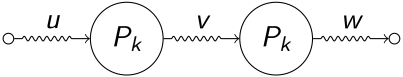

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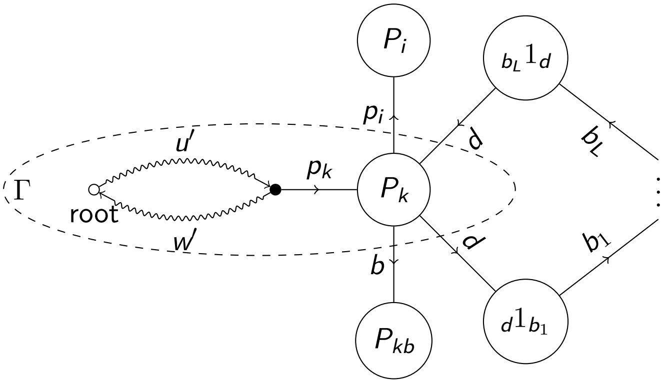

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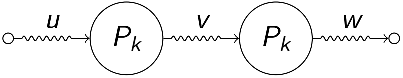

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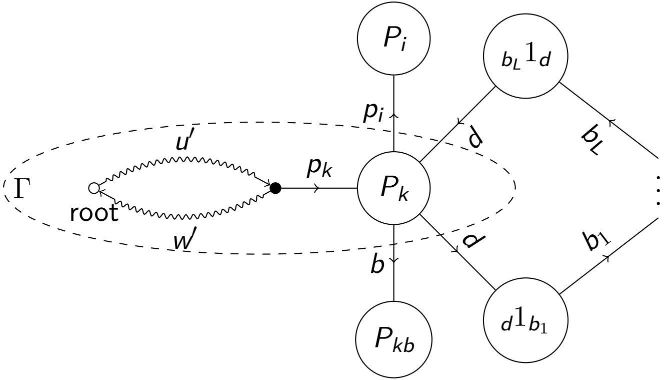

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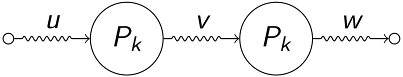

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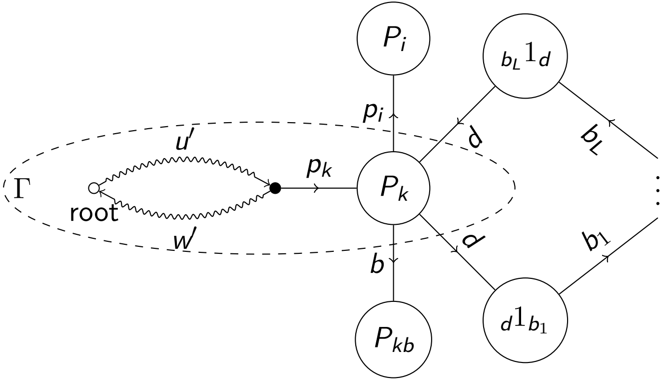

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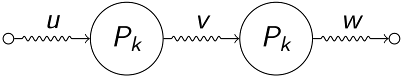

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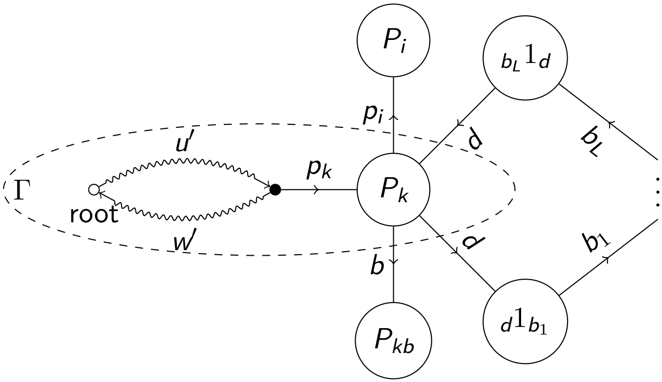

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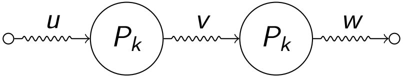

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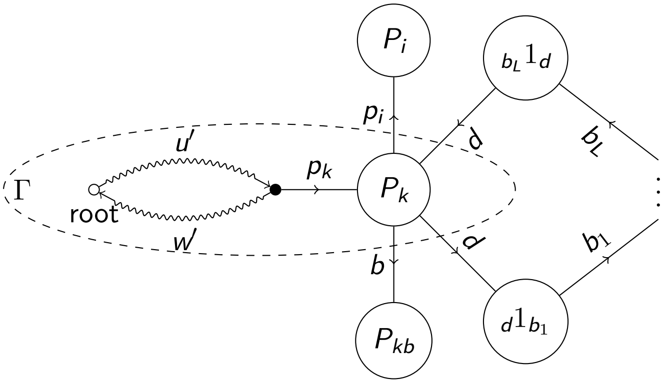

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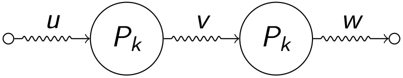

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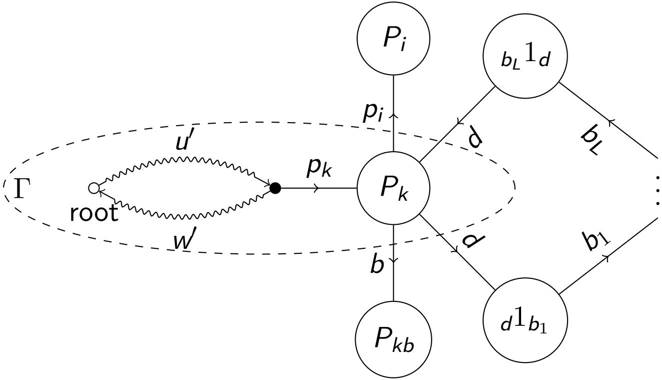

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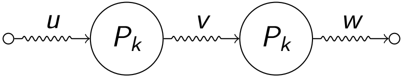

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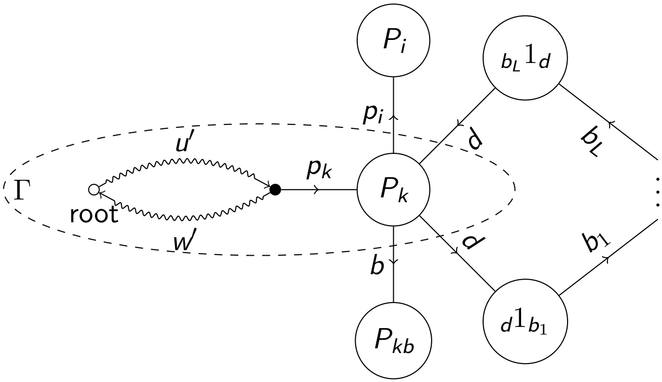

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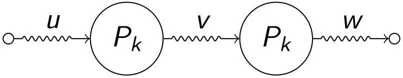

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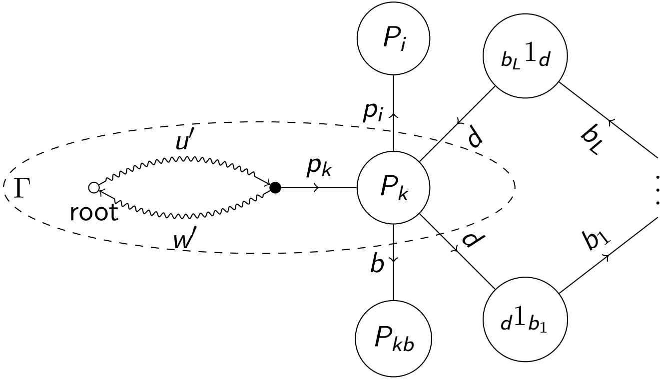

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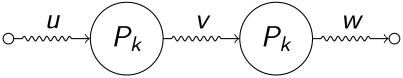

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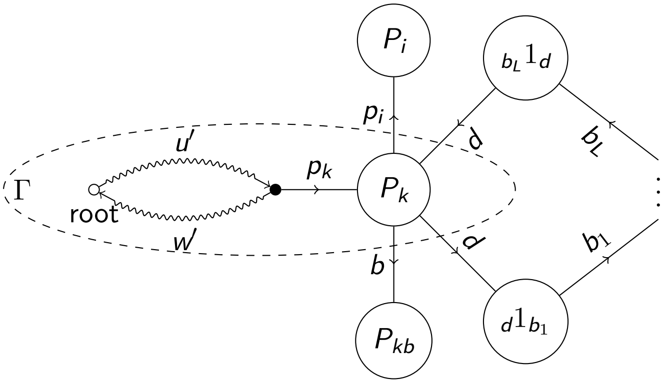

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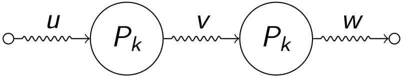

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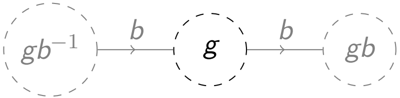

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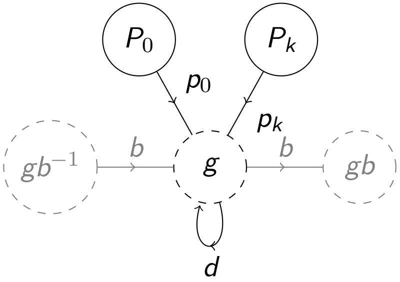

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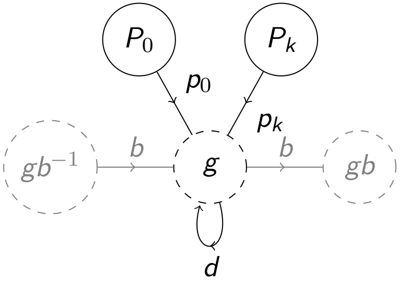

- Then: show the same things as for $S\Gamma(1)$...

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- What about recursively presented groups?

Thank you!