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Maximal Subgroups of Special Inverse Monoids II

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joint work with
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Special Monoids and the Inverse E-Unitary Case

Theorem (Malheiro; 2005)

M: special (ordinary, non-inverse) monoid
All maximal subgroups are isomorphic to the group of units: $\forall e \in E(M) : [e]_{\mathcal{H}} \simeq [1]_{\mathcal{H}}$

Theorem (Gray, Kambites; arXiv 2023)

M: E-unitary special inverse monoid
All maximal subgroups virtually embed into the group of units:

$$\forall e \in E(M) : [e]_{\mathcal{H}} \geq_{f.i.} H \hookrightarrow [1]_{\mathcal{H}}$$

In both cases: The group of units "dominates" the maximal subgroups.

What about arbitrary (non-E-unitary) inverse monoids?

Reminder: The Word Problem

Theorem (Ivanov, Margolis, Meakin; 2001)

The word problem for one-relator monoids reduces to the word problem for special one-relator inverse monoid. ⇜ these are generally not E-unitary!

The Non-E-Unitary Case

Theorem (Gray, Kambites, W.; WIP)

G: finitely presented group $\implies \exists M_G$: special inverse monoid s. t.

1. *the group of units $[1]_{\mathcal{H}}$ is trivial and*
2. *G is the maximal subgroup at some idempotent e (i. e. $[e]_{\mathcal{H}} \simeq G$)*

Theorem (Stephen; 1990)

e: idempotent in an inverse monoid $M \implies \operatorname{Aut} S\Gamma(e) \simeq [e]_{\mathcal{H}}$

Thus: We need to construct M_G such that

1. $S\Gamma(1)$ has trivial automorphism group and
2. the automorphism group of $S\Gamma(e)$ is G.

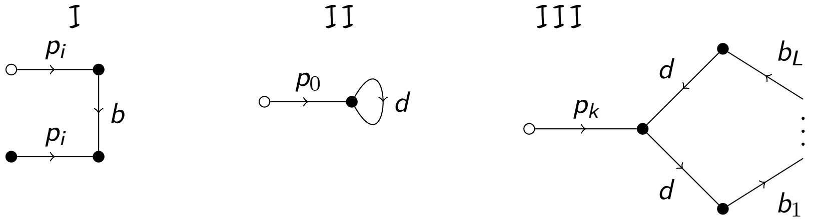
Construction

$G = \mathrm{Mon}\langle B \mid r_1 = \cdots = r_R = 1 \rangle$: any finitely presented group with $r_k \in B^+$

e.g.: $\mathrm{Mon}\langle b, c \mid bc = cb = 1 \rangle \simeq \mathbb{Z}$

$$M_G = \mathrm{Inv}\left\langle B, p_0, p_1, \dots, p_R, d \;\middle|\; \begin{array}{l} \text{I: } p_i b p_i^{-1} p_i b^{-1} p_i^{-1} = 1 \text{ for all } b \in B, i \in \{0, \dots, R\}, \\ \text{II: } p_0 d p_0^{-1} = 1, \\ \text{III: } p_k d r_k d p_k^{-1} = 1 \text{ for all } k \in \{1, \dots, R\} \end{array} \right\rangle$$

Graphically:

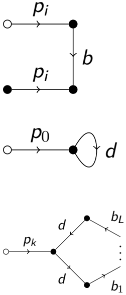


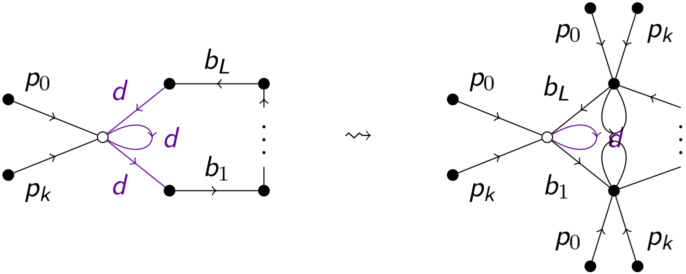
where $r_k = b_1 \dots b_L$

Idea of the Construction

- Consider the idempotent $e = p_0^{-1} p_0 \prod_{k=1}^{R} p_k^{-1} p_k$:

Relations:



- 

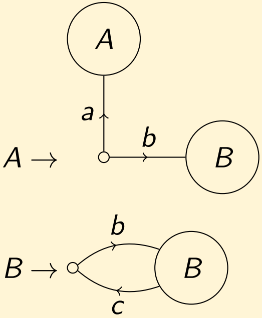
- We attach a "decorated" loop labeled by a relator.
- ⇝ "decorated" Cayley graph of G
- It turns out: the additional parts yield no additional automorphisms!
- How can we make this formal? We need an appropriate description!

A Grammar to Describe Tree-Like Graphs

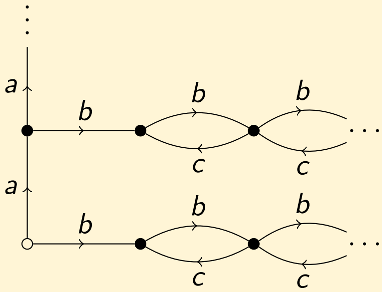
Example

$V = \{A, B\}$: set of nonterminals $\quad \Sigma = \{a, b\}$: edge labels

Rules:

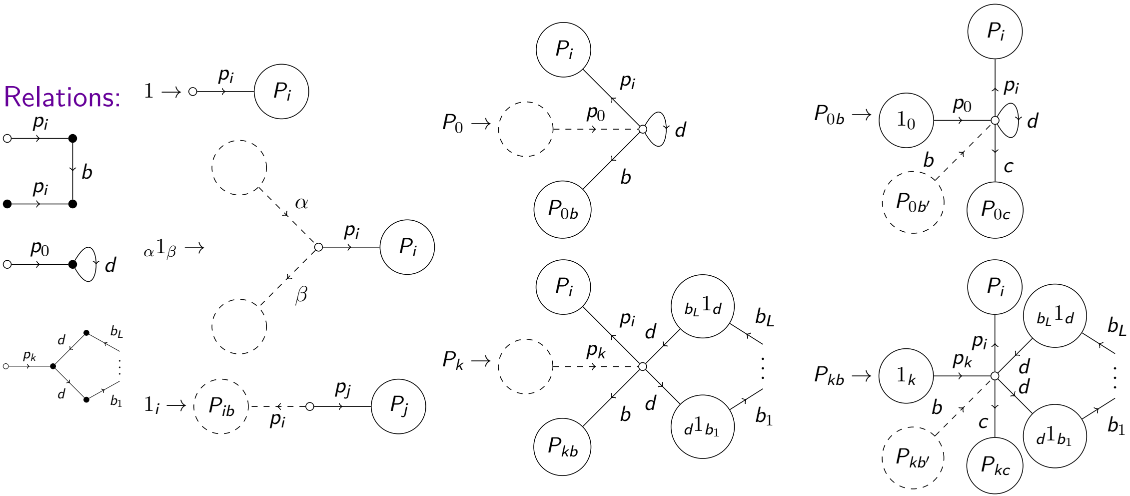


Generated Graph: vs "intermediate graphs"



A Grammar for $S\Gamma(1)$

Relations:



How do we Know that this Indeed Generates $S\Gamma(1)$?

Theorem (Stephen; 1990)

$M = \mathrm{Inv}\langle A \mid \lambda_i = 1 \rangle$ $\quad$ $e \in A^{\pm *}$ *with* $e^2 = e$ *in* M $\quad$ $\Gamma : A^{\pm 1}$*-labeled directed graph with root* q

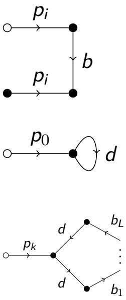
$\Gamma \simeq S\Gamma(e) \iff$

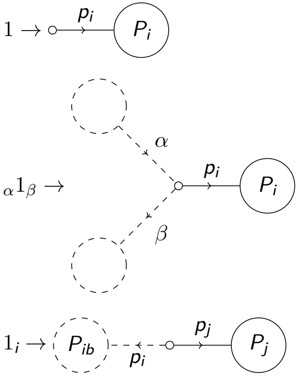
1. Γ *is symmetric, strongly connected, deterministic* $\rightsquigarrow$ *check neighborhoods*
2. $e \in \mathscr{L}(q, q)$ $\rightsquigarrow$ *trivial for* $e = 1$
3. $\forall p \in \Gamma : \lambda_i \in \mathscr{L}(p, p)$ $\rightsquigarrow$ *check extended neighborhoods*
4. $\mathscr{L}(q, q) \subseteq \mathscr{U}(e) = \{u \in A^{\pm *} \mid u \geq e \text{ in } e\}$
   $\rightsquigarrow$ This is the tricky part!

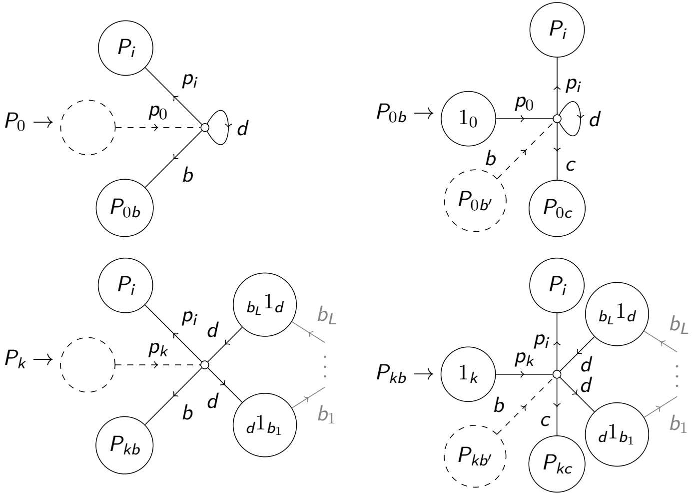
In our grammar: the neighborhood is fully determined by the nonterminal!
This neighborhood characterization also helps us to show that there are no automorphism.
Note: We only have to show that the root must be mapped to itself.

Again: The Grammar for $S\Gamma(1)$

Relations:







deterministic ✓

relations readable ✓

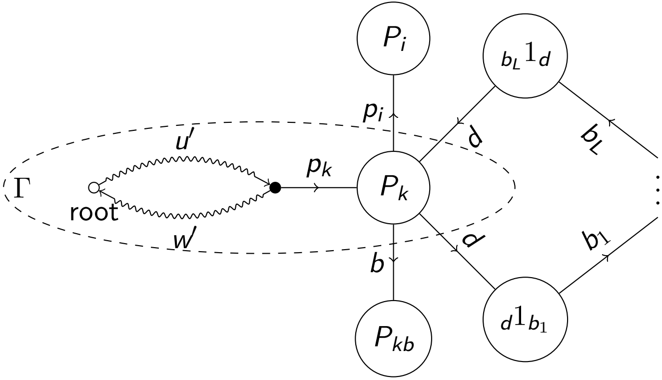
images of the root ✓

only 1 or $1_i \rightsquigarrow$ "$p_i b^{-1}$"

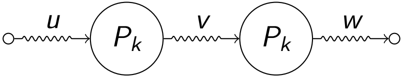
Upward Closure

- It remains to show: $\mathscr{L}(q, q) \subseteq \mathscr{U}(1) = \{u \in A^{\pm *} \mid u \geq 1 \text{ in } e\}$
- Formally, we define the generated graph Γ^* as the direct limit of the intermediate graphs Γ.
- Thus: It suffice to show the inclusion for all intermediate graphs!
- This allows for an inductive argument:
  Assume: Γ turns into Γ' in one step and $\mathscr{L}(\Gamma) \subseteq \mathscr{U}(1)$
  To show: $\mathscr{L}(\Gamma') \subseteq \mathscr{U}(1)$

Sketch of the Induction



- Let x label a circle at the root.
- If it lies completely in Γ, we have $x \in \mathscr{U}(1)$ by induction.
- Otherwise, factorize it at P_k:

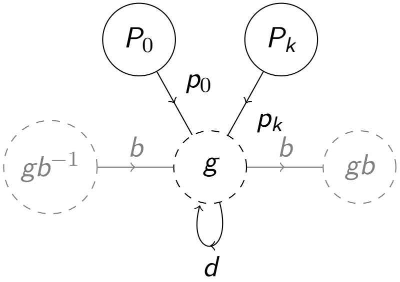


- W.l.o.g.: no other P_k visits
- We know: $u = u' p_k$ and $w = p_k^{-1} w'$
- Options for v
  - $v = p_i p_i^{-1}$
  - $v = bb^{-1}$
  - $v = db_1 \ldots b_L d$
  - $v = db_1 \ldots b_i b_i^{-1} \ldots b_1^{-1} d^{-1}$

- $x = uvw = u' \underbrace{p_k db_1 \ldots b_l p_k^{-1}}_{=1} w' = u' w' \in \mathscr{U}(1)$

What About the Grammar for $S\Gamma(e)$?

- We re-use the grammar for $S\Gamma(1)$.
- This time we don't start with a single node but with the Cayley graph of G.
- Add the appropriate "decorations" to each node:



- Then: show the same things as for $S\Gamma(1)$...

Outlook

- We can most likely get $G \star \mathbb{Z}$ as the maximal group image.
  Right now: free group of higher rank
- Open: Can we get G as the maximal group image?
- We can probably create any suitable lattice of finitely presented groups.
- What about recursively presented groups?

Thank you!