Maximal Subgroups of Special Inverse Monoids

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joint work with
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Theorem (Magnus' Freiheitssatz; 1930)

The word problem of a one-relator group

Constant:	$G = \mathrm{Grp}\langle A \mid \ell = r \rangle$ one-relator group
Input:	$w \in A^{\pm *}$
Question:	is $w = 1$ in G?

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What about monoids?

Definition (Word Problem of a One-Relator Monoids)

Constant: $M = \mathrm{Mon}\langle A \mid \ell = r \rangle$ one-relator monoid
Input: $u, v \in A^*$
Question: is $u = v$ in M?

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The word problem of M reduces to the word problem of a monoid of the form

$$\mathrm{Mon}\langle a, b \mid aub = ava\rangle \quad \text{or} \quad \mathrm{Mon}\langle a, b \mid aub = a\rangle \quad \text{for } u, v \in \{a, b\}^*.$$

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Theorem (Ivanov, Margolis, Meakin; 2001)

$$\mathrm{Mon}\langle a, b \mid aub = ava\rangle \hookrightarrow \mathrm{Inv}\langle a, b \mid aub(ava)^{-1} = 1\rangle$$
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Thus: Solve the word problem for one-relator special inverse monoids!

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How do we solve the word problem in special inverse monoids?

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The word problem of a special monoid $M = \mathrm{Mon}\langle A \mid \ell_1 = 1, \ldots, \ell_n = 1\rangle$ *is decidable if the word problem of the group of units* $U(M) = [1]_{\mathcal{H}}$ *is decidable.*

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Theorem (Malheiro; 2005)

$M = \mathrm{Mon}\langle A \mid \ell_1 = 1, \ldots, \ell_n = 1 \rangle$ *special monoid*

All maximal subgroups of M are isomorphic to the group of units: $\forall e \in E(M) : [e]_{\mathcal{H}} \simeq [1]_{\mathcal{H}}$

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Is this also true for special inverse monoids?

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What about arbitrary (non-E-unitary) inverse monoids?

These include $\mathrm{Inv}\langle a, b \mid aub(ava)^{-1} = 1\rangle$ and $\mathrm{Inv}\langle a, b \mid auba^{-1} = 1\rangle$ in general.

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There exists a finitely presented special inverse monoid M such that

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2. $\mathrm{Aut}\, S\Gamma(e) = H = \mathrm{Mon}\langle B \mid r_1 = \cdots = r_R = 1\rangle$. e.g.: $\mathrm{Mon}\langle b, c \mid bc = cb = 1\rangle \simeq \mathbb{Z}$

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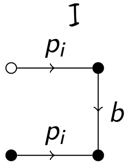

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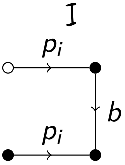

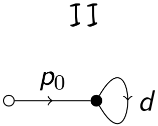

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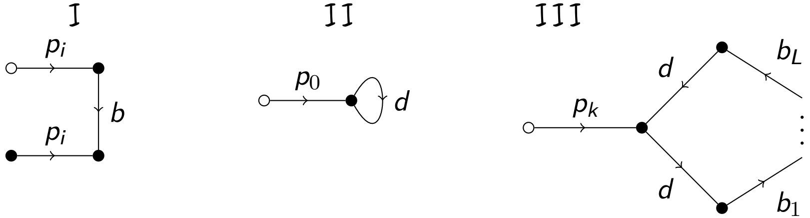
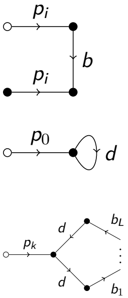


where $r_k = b_1 \ldots b_L$

Idea of the Construction

Relations:



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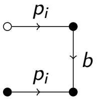

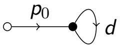

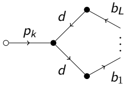

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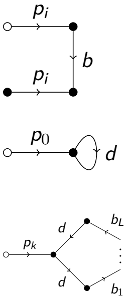
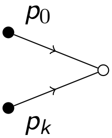


- 

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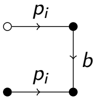
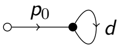
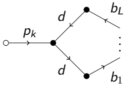
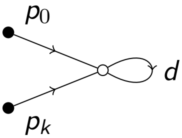






- 

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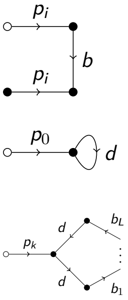
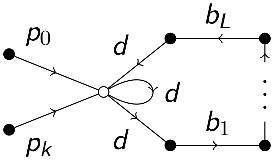


- 

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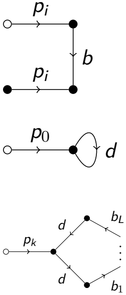
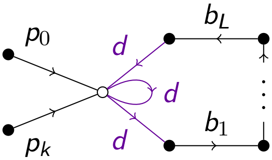


- 

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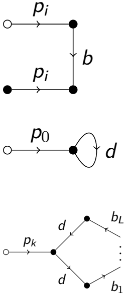
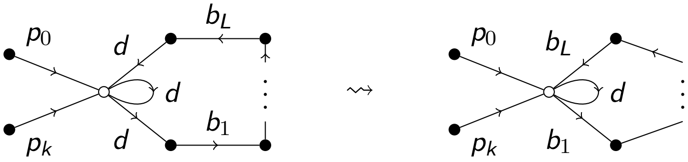


- 

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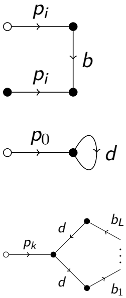
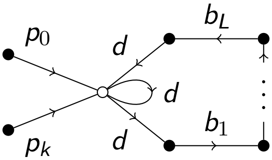


- 

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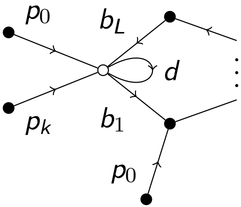

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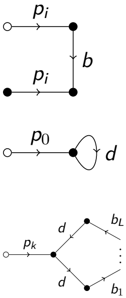
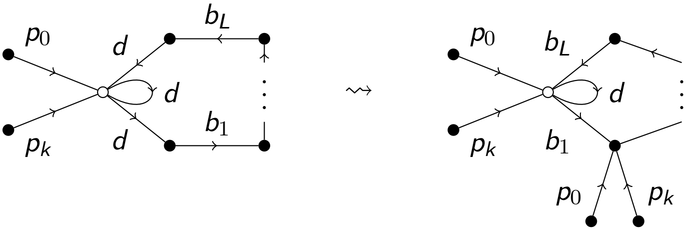
Relations:



- 

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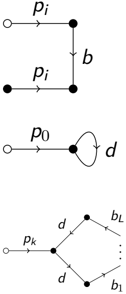
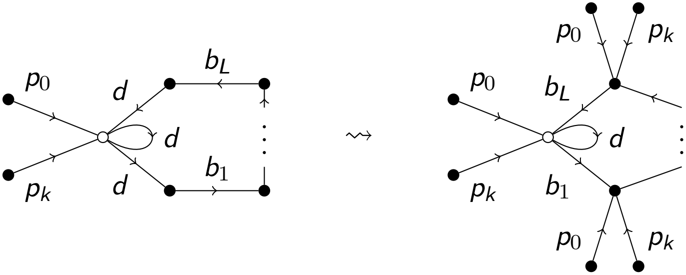


- 

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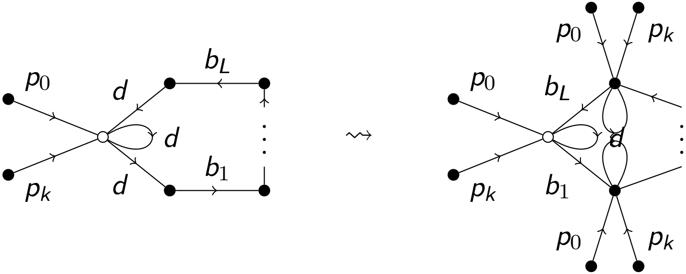
- 

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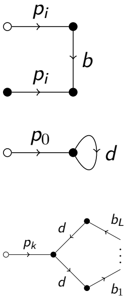

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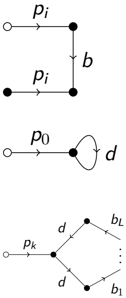
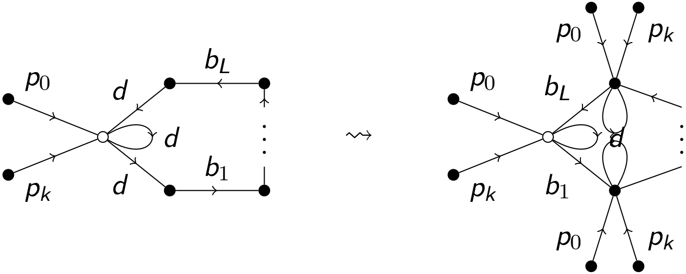


- 

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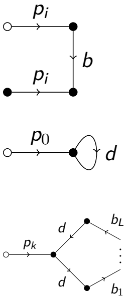
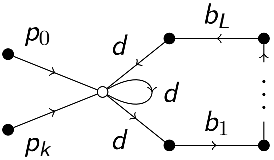
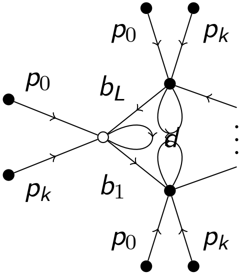


- 

$\rightsquigarrow$


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- $\rightsquigarrow$ "decorated" Cayley graph of H

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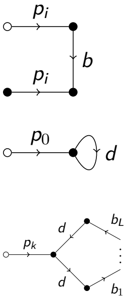
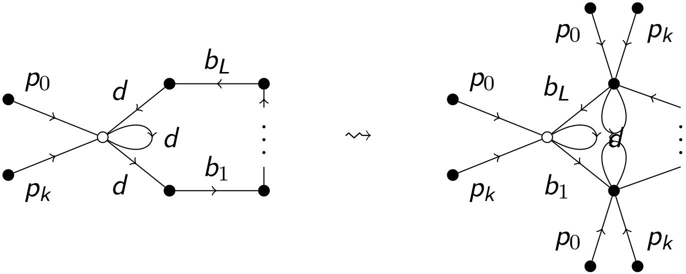


- 

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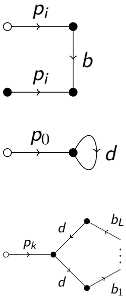
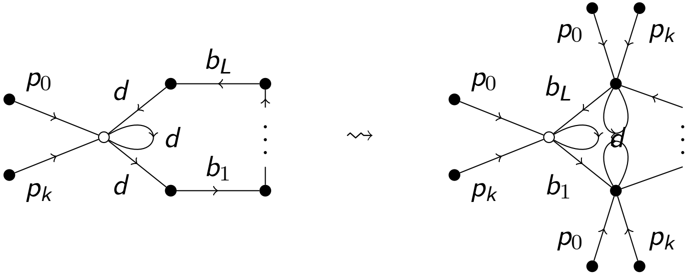


- 

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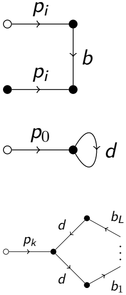
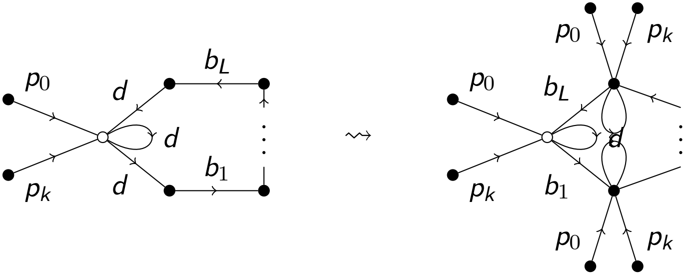


- 

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- It turns out: the additional parts yield no additional automorphisms!
- How can we make this formal? We need an appropriate description!

A Grammar to Describe Tree-Like Graphs

Example

A Grammar to Describe Tree-Like Graphs

Example

$V = \{A, B\}$: set of nonterminals

A Grammar to Describe Tree-Like Graphs

Example

$V = \{A, B\}$: set of nonterminals $\quad \Sigma = \{a, b\}$: edge labels

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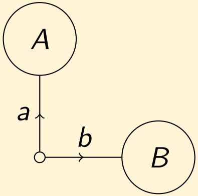
$A \to$ 

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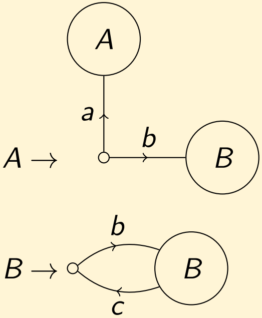

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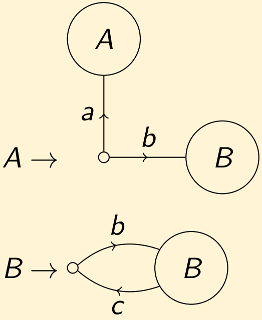

Generated Graph:

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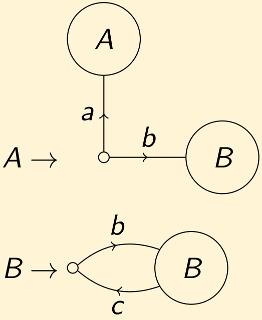

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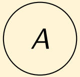

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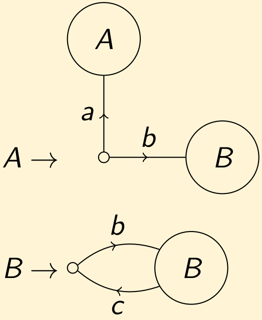

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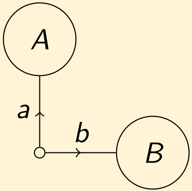

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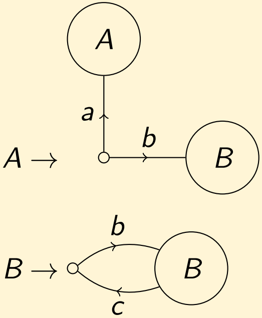

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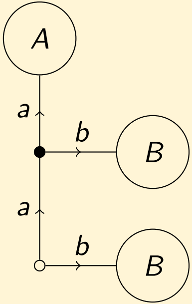

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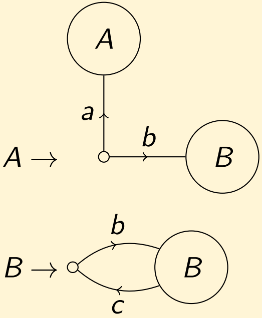

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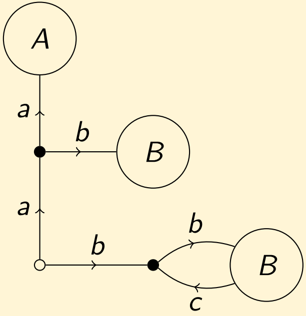

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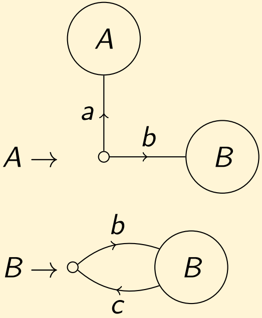

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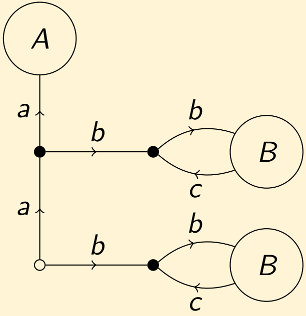

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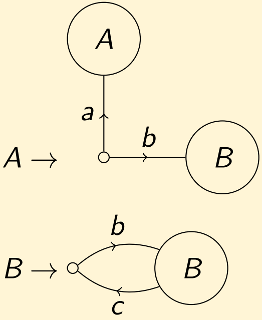

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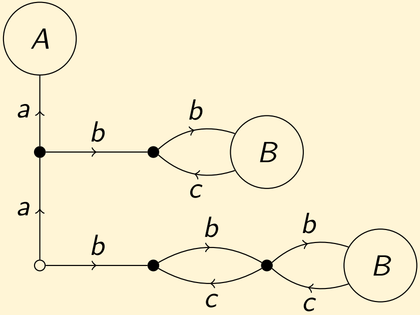

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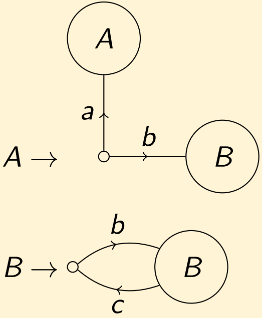

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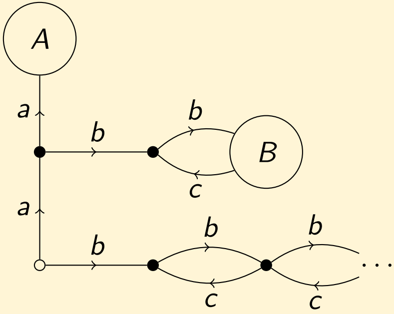

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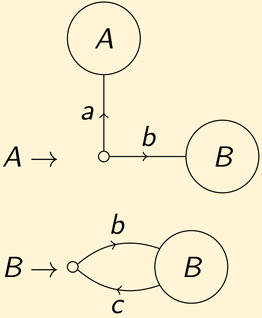

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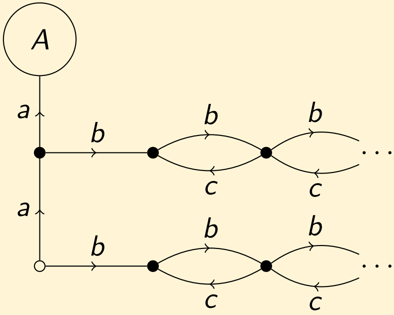

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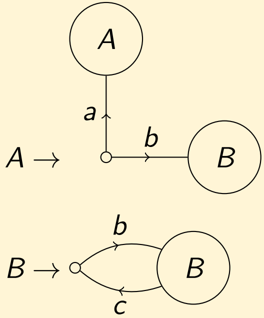
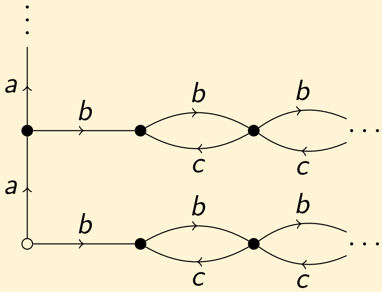


Generated Graph: vs "intermediate graphs"



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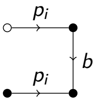
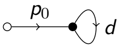
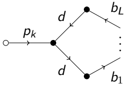
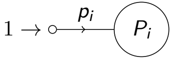


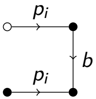


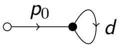


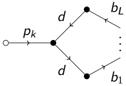
A Grammar for $S\Gamma(1)$

Relations:









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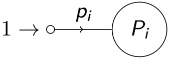

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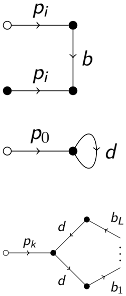

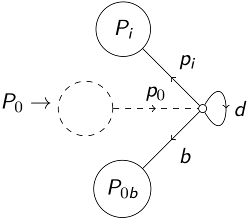

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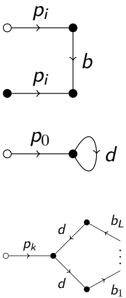

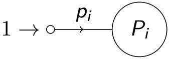

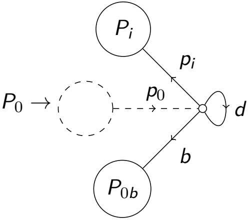

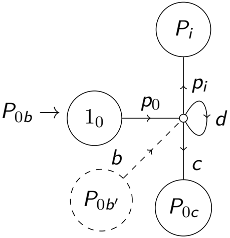

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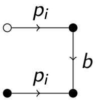
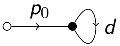
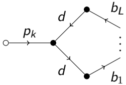
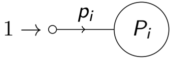
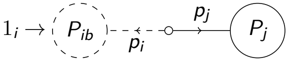
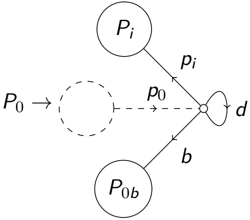

pi
b
pi

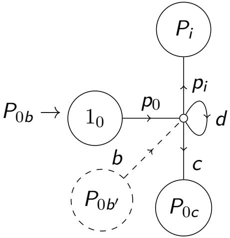

p0
d


pk
d
bL
d
b1

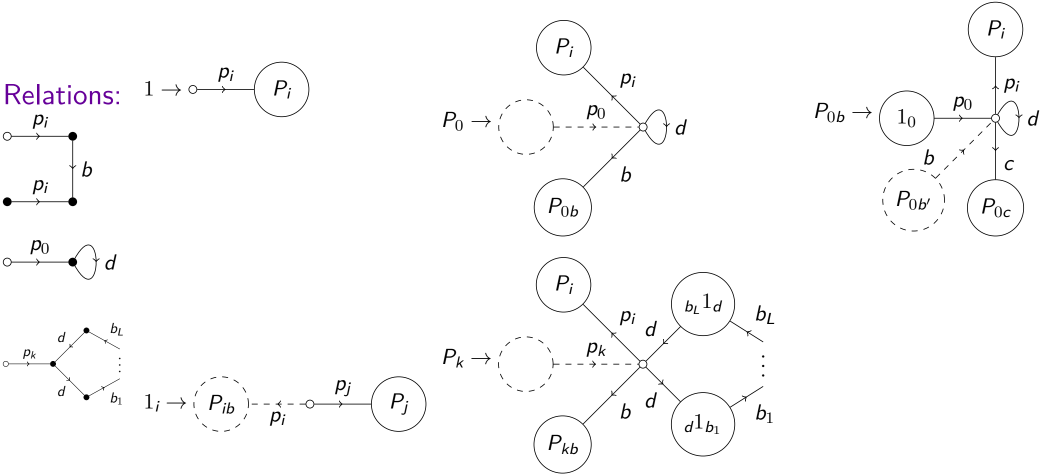

1 →
pi
Pi


1i →
Pib
pi
pj
Pj


P0 →
Pi
pi
p0
d
b
P0b


P0b →
Pi
1o
p0
pi
d
b
c
P0b′
P0c

A Grammar for $S\Gamma(1)$



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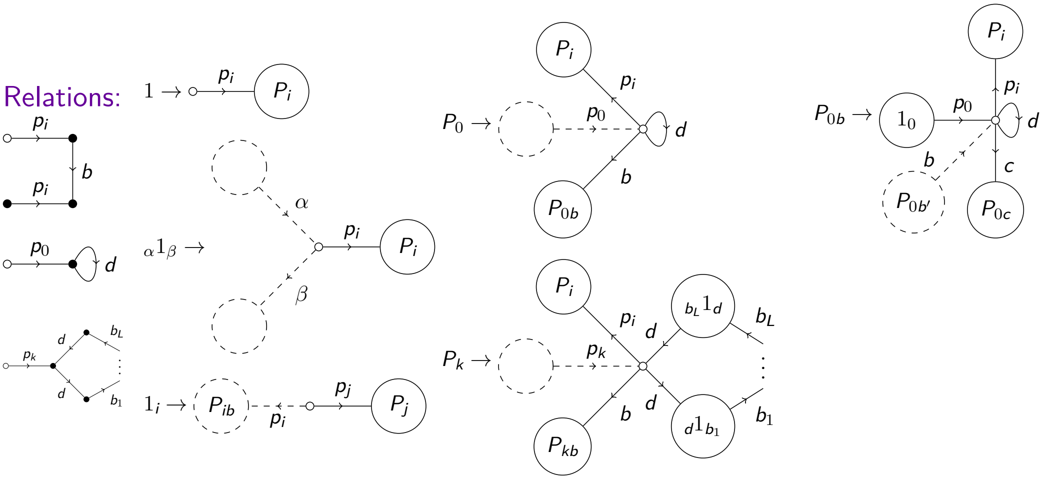

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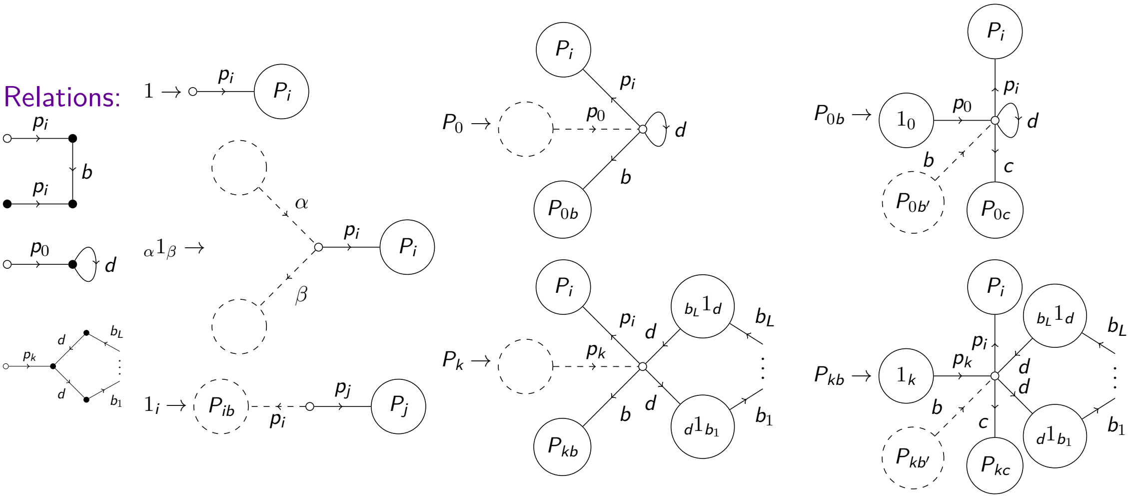

How do we Know that this Indeed Generates $S\Gamma(1)$?

Theorem (Stephen; 1990)

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$M = \mathrm{Inv}\langle A \mid \lambda_i = 1 \rangle$

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Theorem (Stephen; 1990)

$M = \mathrm{Inv}\langle A \mid \lambda_i = 1 \rangle$ $\quad e \in A^{\pm *}$ *with* $e^2 = e$ *in* M

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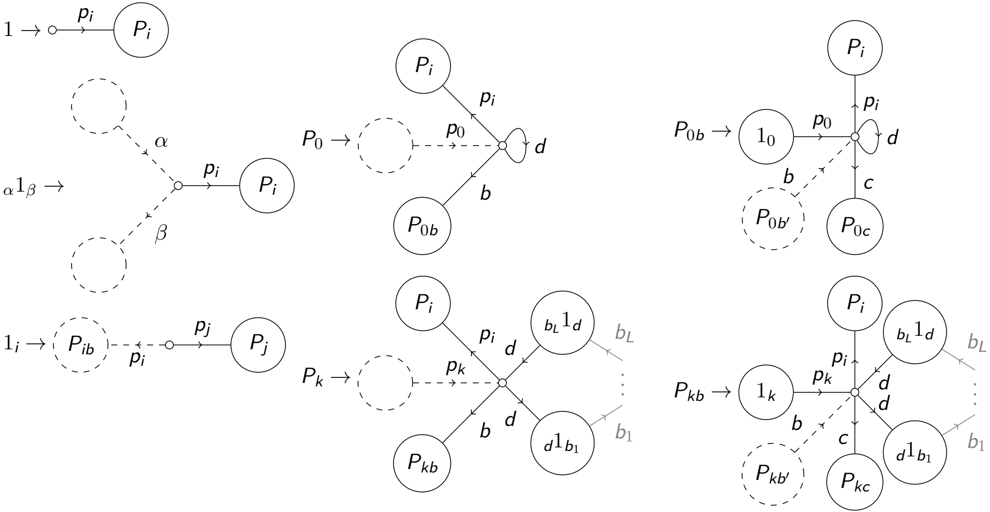

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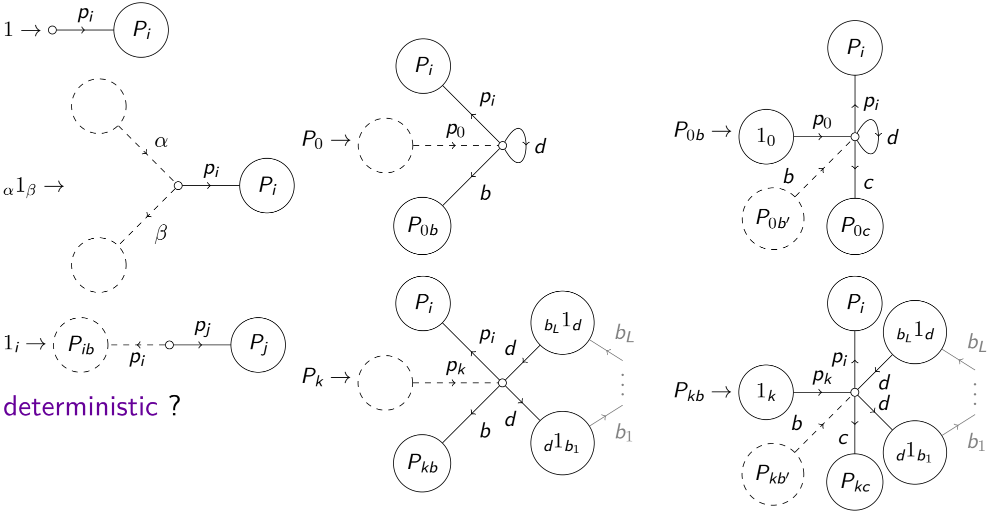

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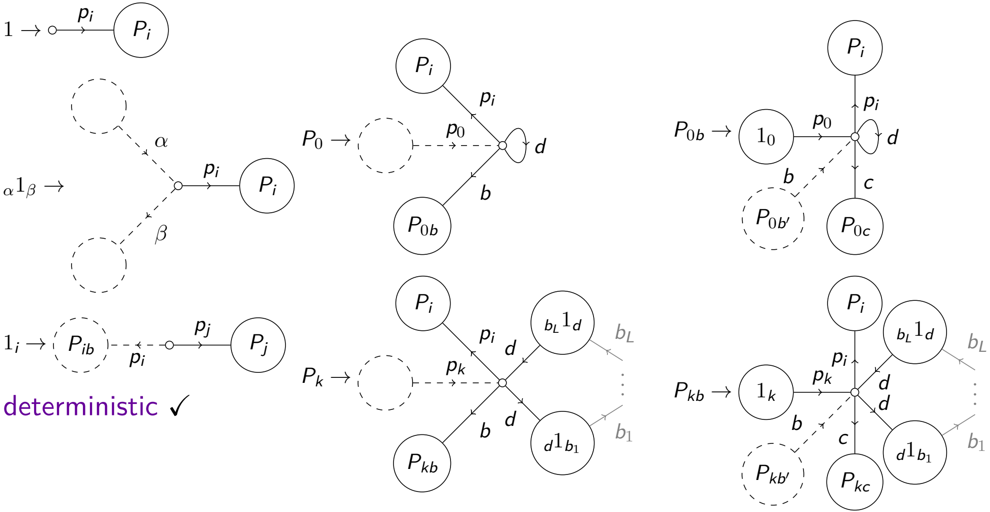

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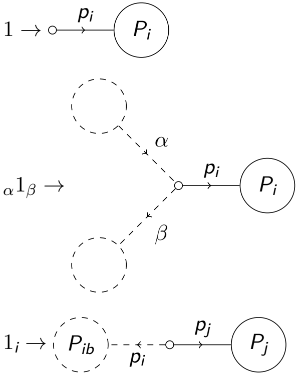

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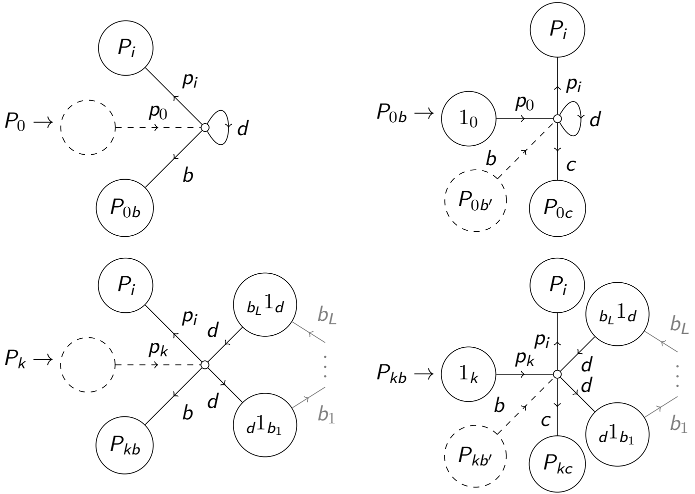

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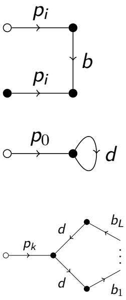

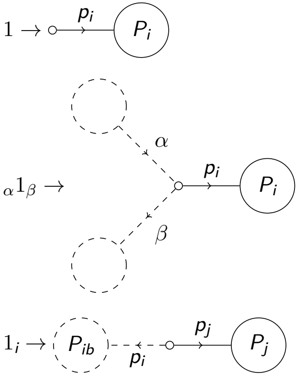

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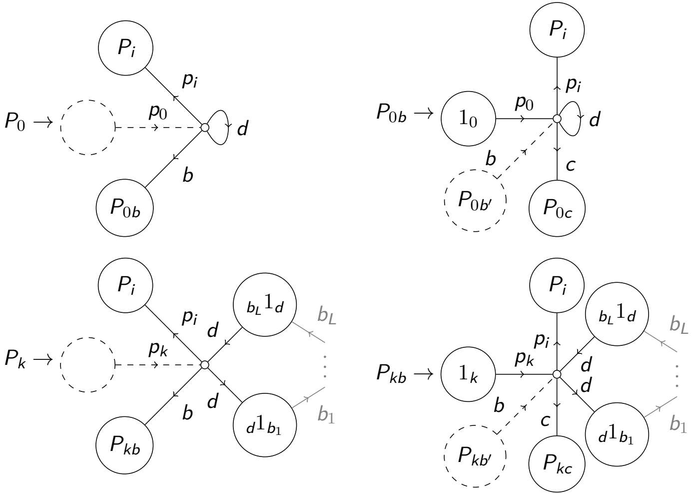

Again: The Grammar for $\mathcal{ST}(1)$

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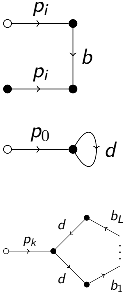

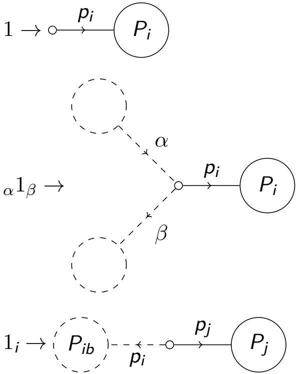

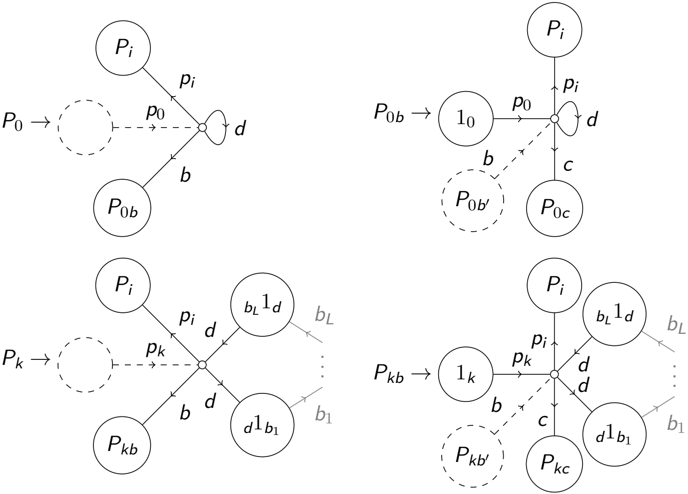

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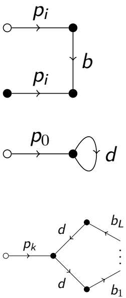

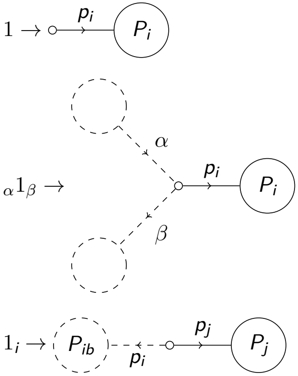

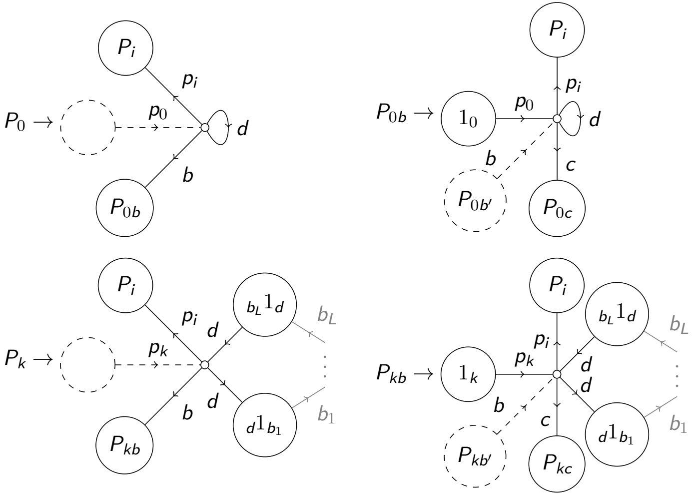

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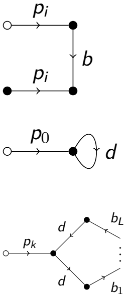

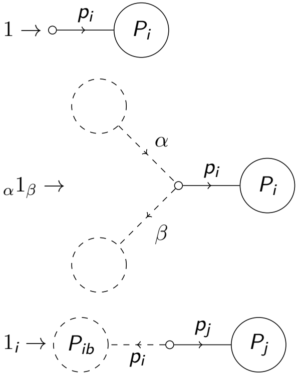

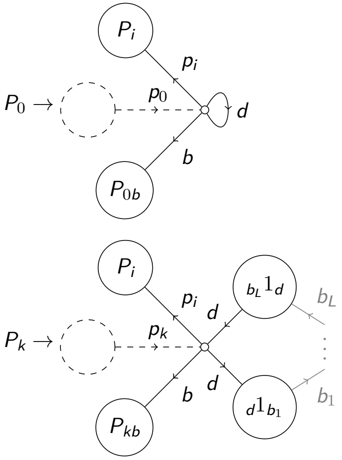

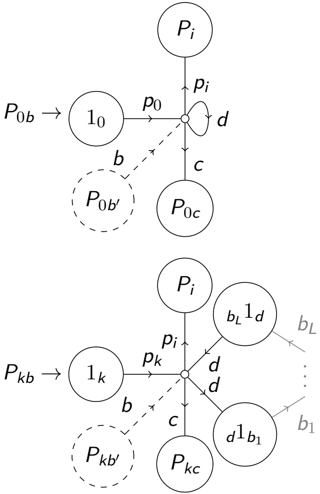

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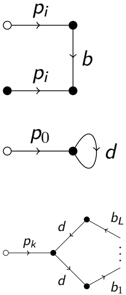

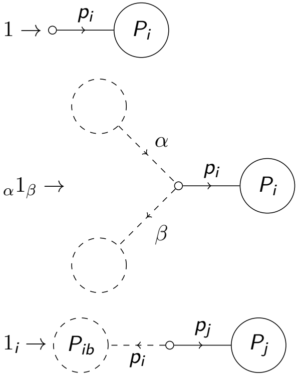

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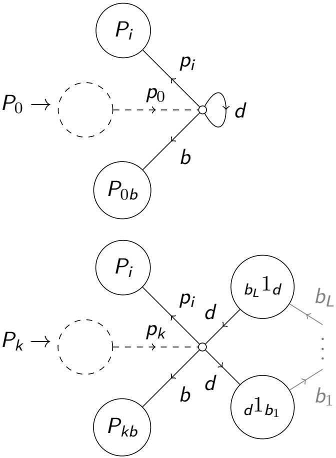

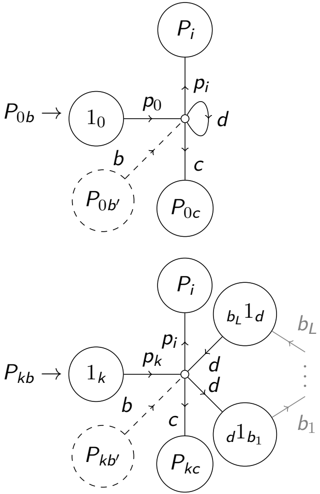

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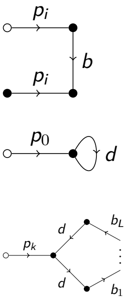

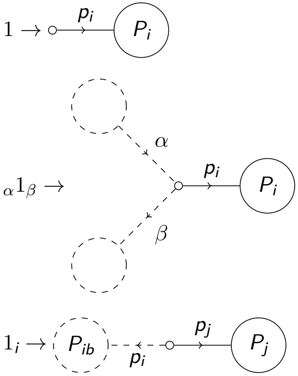

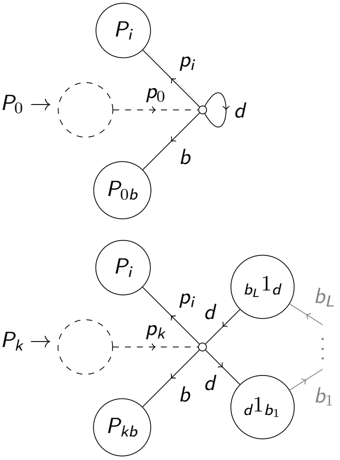

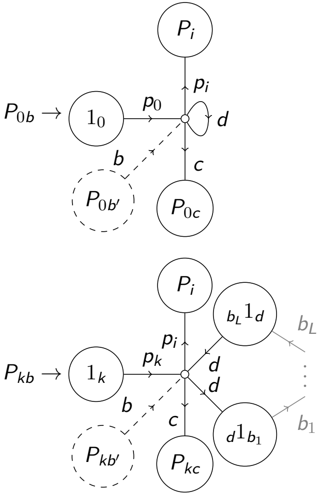

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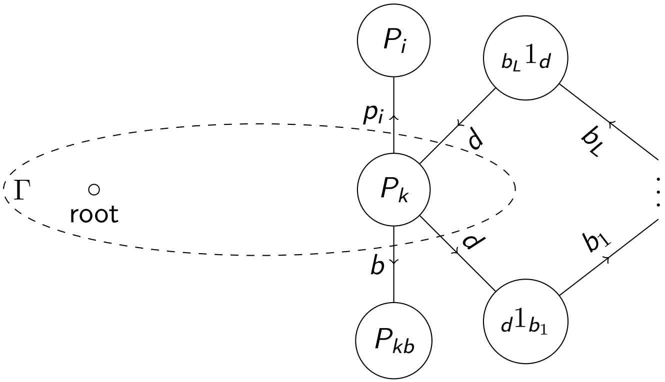

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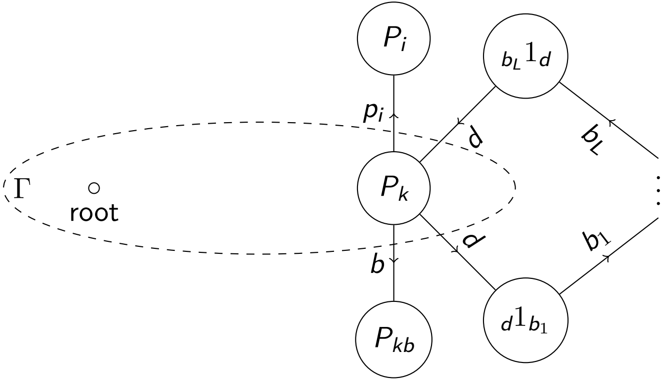

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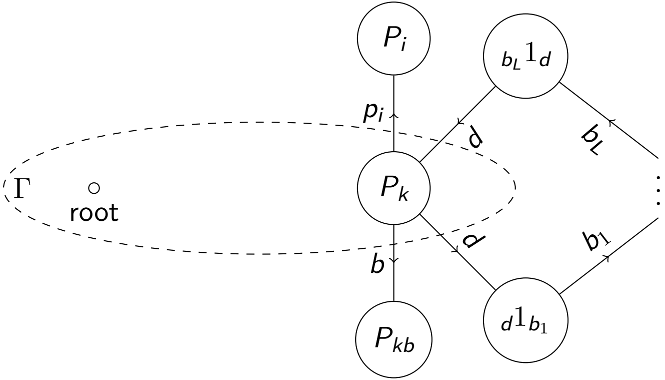

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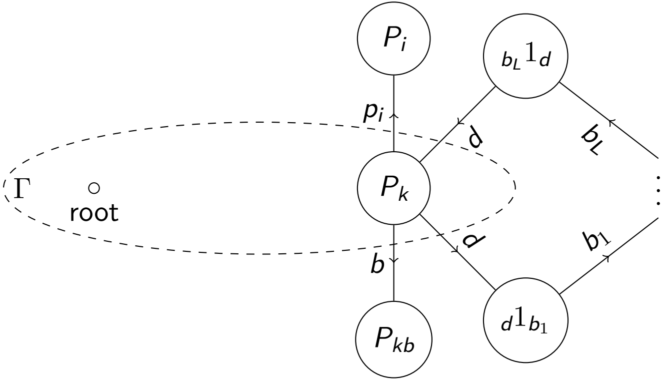

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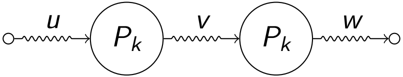

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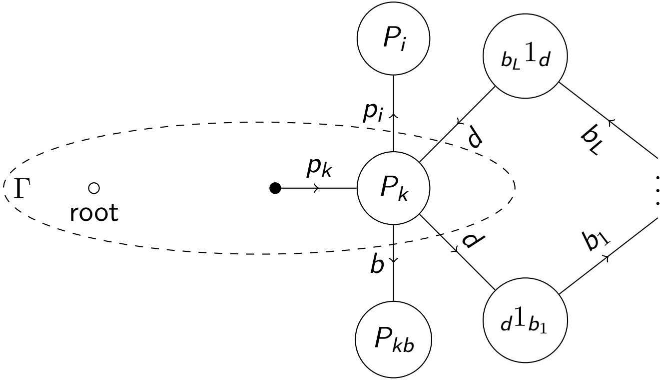

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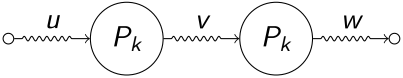

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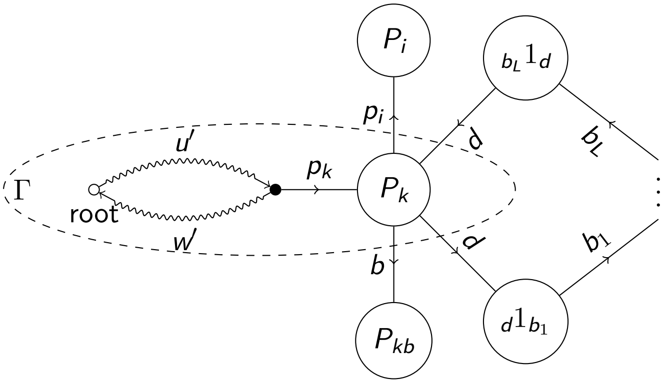

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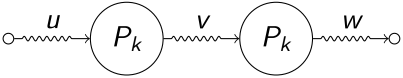

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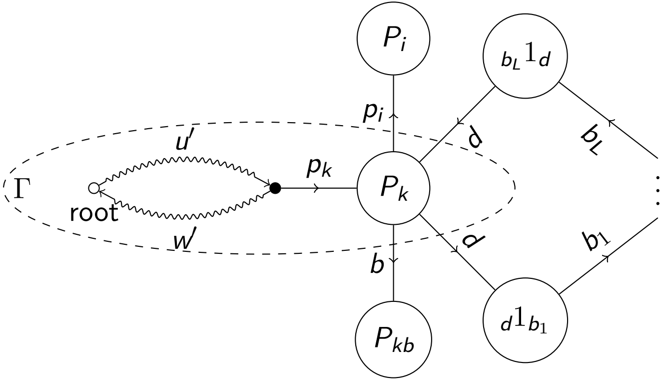

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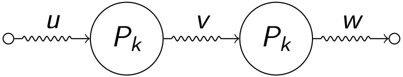

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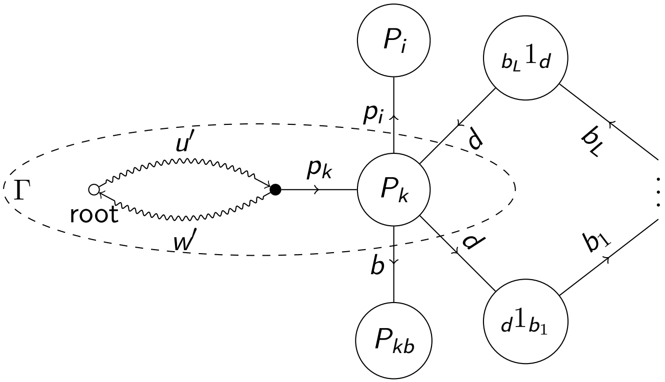

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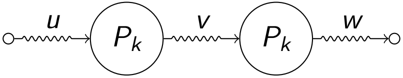

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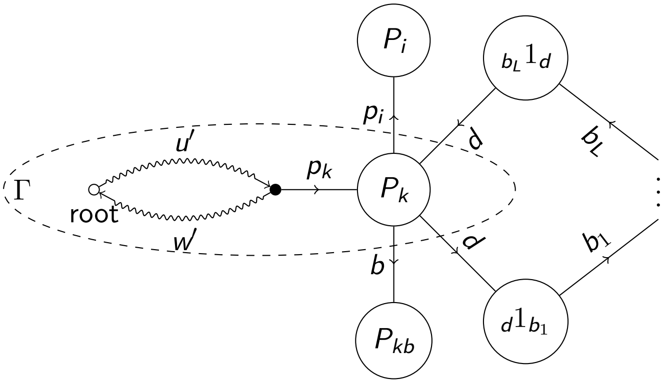

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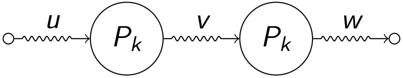

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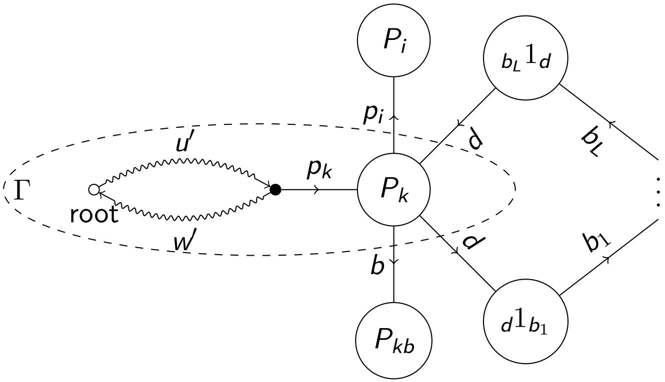
Sketch of the Induction



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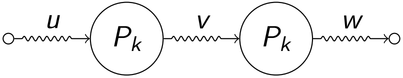

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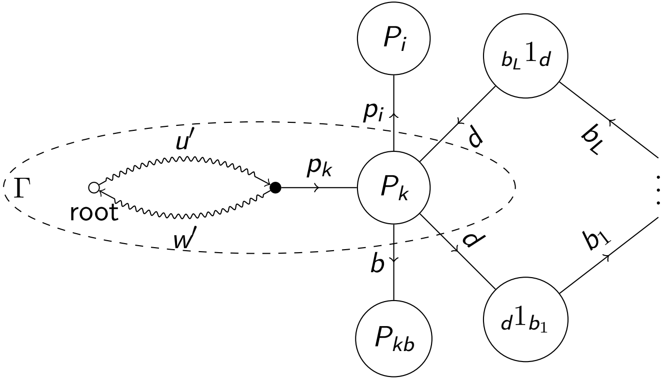

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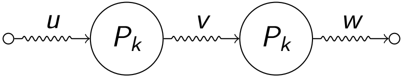

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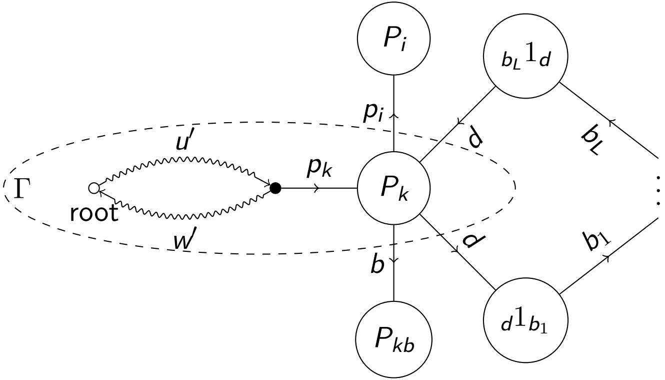

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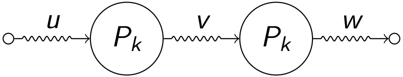

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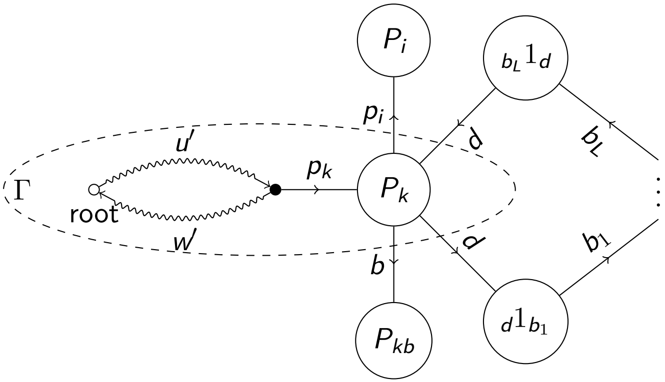

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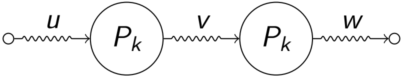

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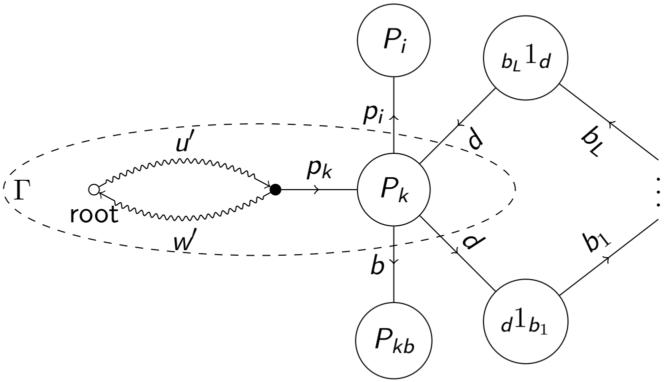

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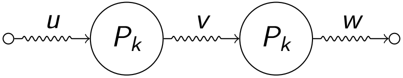

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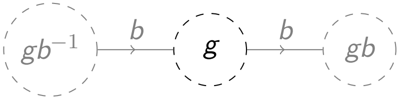

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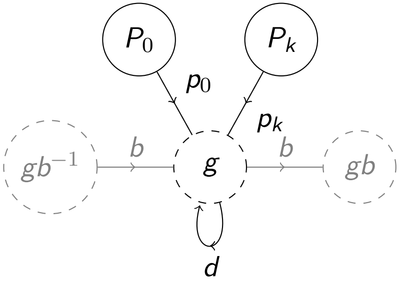

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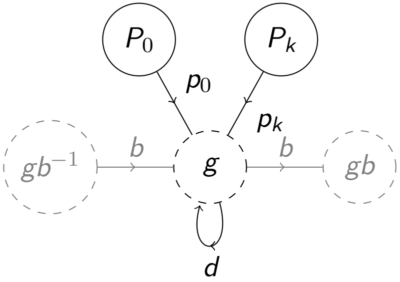

- Then: show the same things as for $S\Gamma(1)$...

Thank you!