Maximal Subgroups of Special Inverse Monoids

Jan Philipp **Wächter**

Department of Mathematics
University of Manchester

joint work with
Robert **Gray** and Mark **Kambites**

This research was supported by EPSRC

22 April 2026

The Word Problem for One-Relator Monoids

Theorem (Magnus' Freiheitssatz; 1930)

The word problem of a one-relator group

Constant: $G = \mathrm{Grp}\langle A \mid \ell = r \rangle$ one-relator group
Input: $w \in A^{\pm *}$
Question: is $w = 1$ in G?

is decidable.

What about monoids?

Definition (Word Problem of a One-Relator Monoids)

Constant: $M = \mathrm{Mon}\langle A \mid \ell = r \rangle$ one-relator monoid
Input: $u, v \in A^{*}$
Question: is $u = v$ in M?

Remaining Cases

Theorem (Adian, Oganesian; 1987)

$M = \mathrm{Mon}\langle A \mid \ell = r \rangle$ one-relator monoid Then:
The word problem of M reduces to the word problem of a monoid of the form

$$\mathrm{Mon}\langle a, b \mid aub = ava \rangle \quad \text{or} \quad \mathrm{Mon}\langle a, b \mid aub = a \rangle \quad \text{for } u, v \in \{a, b\}^*.$$

Theorem (Ivanov, Margolis, Meakin; 2001)

$$\mathrm{Mon}\langle a, b \mid aub = ava \rangle \hookrightarrow \mathrm{Inv}\langle a, b \mid aub(ava)^{-1} = 1 \rangle$$
$$\mathrm{Mon}\langle a, b \mid aub = a \rangle \hookrightarrow \mathrm{Inv}\langle a, b \mid auba^{-1} = 1 \rangle$$

Thus: Solve the word problem for one-relator special inverse monoids!

The Word Problem for One-Relator Special Inverse Monoids

The bad news:

Theorem (Gray; 2019)

There is an E-unitary one-relator special inverse monoid with undecidable word problem.

The good news:

The constructed inverse monoid is neither of the form

$$\mathrm{Inv}\langle a, b \mid aub(ava)^{-1} = 1\rangle \text{ nor } \mathrm{Inv}\langle a, b \mid auba^{-1} = 1\rangle.$$

How do we solve the word problem in special inverse monoids?

Ordinary, Non-Inverse Special Monoids

Theorem (Adian; 1960)

The word problem of any $M = \mathrm{Mon}\langle A \mid \ell = 1\rangle$ *is decidable.*

Theorem (Makanin; 1966)

The word problem of a special monoid $M = \mathrm{Mon}\langle A \mid \ell_1 = 1, \dots, \ell_n = 1\rangle$ *is decidable if the word problem of the group of units* $U(M) = [1]_{\mathcal{H}}$ *is decidable.*

Theorem (Malheiro; 2005)

$M = \mathrm{Mon}\langle A \mid \ell_1 = 1, \dots, \ell_n = 1\rangle$ *special monoid*
All maximal subgroups of M are isomorphic to the group of units: $\forall e \in E(M) : [e]_{\mathcal{H}} \simeq [1]_{\mathcal{H}}$

Special monoids are "dominated" by their group of units.
Is this also true for special inverse monoids?

The E-Unitary Case

Theorem (Gray, Kambites; 2025)

M: E-unitary special inverse monoid
All maximal subgroups of M virtually embed into the group of units:

$$\forall e \in E(M) : [e]_{\mathcal{H}} \geq_{f.\,i.} H \hookrightarrow [1]_{\mathcal{H}}$$

The group of units still "virtually dominates" the monoid.

What about arbitrary (non-E-unitary) inverse monoids?

These include $\mathrm{Inv}\langle a, b \mid aub(ava)^{-1} = 1\rangle$ and $\mathrm{Inv}\langle a, b \mid auba^{-1} = 1\rangle$ in general.

The Non-E-Unitary Case

Theorem (Gray, Kambites, W.; WIP)

G, H: finitely presented groups

There exists a finitely presented special inverse monoid M such that

1. *the group of units $U(M)$ is ~~G~~ and* trivial
2. *the maximal subgroup at some idempotent e is H (i. e. $[e]_{\mathcal{H}} \simeq G$)*

Theorem (Stephen; 1990)

e: idempotent in an inverse monoid $M \implies \mathrm{Aut}\, S\Gamma(e) \simeq [e]_{\mathcal{H}}$

Thus: We need to construct M such that

1. $\mathrm{Aut}\, S\Gamma(1)$ ~~$= G$~~ is trivial and
2. $\mathrm{Aut}\, S\Gamma(e) = H = \mathrm{Mon}\langle B \mid r_1 = \cdots = r_R = 1\rangle$.

e.g.: $\mathrm{Mon}\langle b, c \mid bc = cb = 1\rangle \simeq \mathbb{Z}$

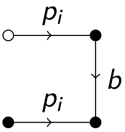
Construction

$H = \mathrm{Mon}\langle B \mid r_1 = \cdots = r_R = 1 \rangle$: any finitely presented group with $r_k \in B^+$

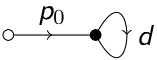
$$
\begin{aligned}
M = \mathrm{Inv}\left\langle B, p_0, p_1, \ldots, p_R, d \;\middle|\; \right. & \mathrm{I}\colon p_i b p_i^{-1} p_i b^{-1} p_i^{-1} = 1 \text{ for all } b \in B, i \in \{0, \ldots, R\}, \\
& \mathrm{II}\colon p_0 d p_0^{-1} = 1, \\
& \left. \mathrm{III}\colon p_k d r_k d p_k^{-1} = 1 \text{ for all } k \in \{1, \ldots, R\} \right\rangle
\end{aligned}
$$

Graphically:

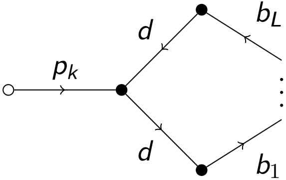
I



II



III

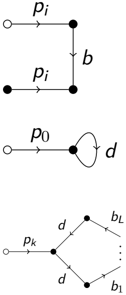


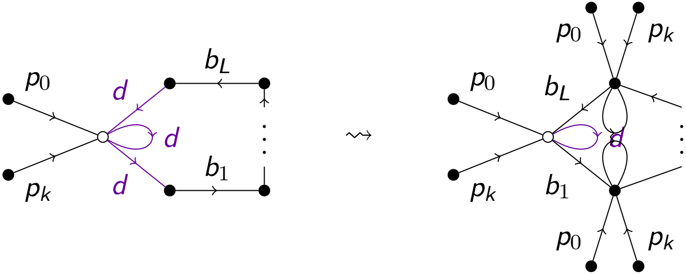
where $r_k = b_1 \ldots b_L$

Idea of the Construction

- Consider the idempotent $e = p_0^{-1} p_0 \prod_{k=1}^{R} p_k^{-1} p_k$:

Relations:



- 

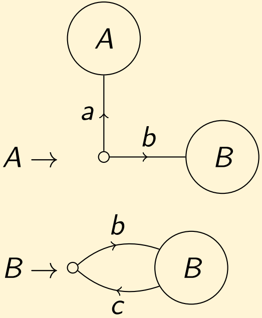
- We attach a "decorated" loop labeled by a relator.
- $\rightsquigarrow$ "decorated" Cayley graph of H
- It turns out: the additional parts yield no additional automorphisms!
- How can we make this formal? We need an appropriate description!

A Grammar to Describe Tree-Like Graphs

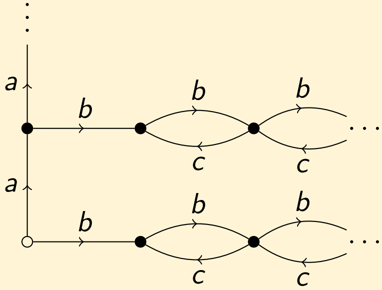
Example

$V = \{A, B\}$: set of nonterminals $\Sigma = \{a, b\}$: edge labels

Rules:

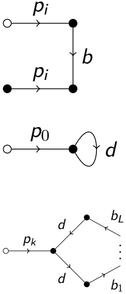


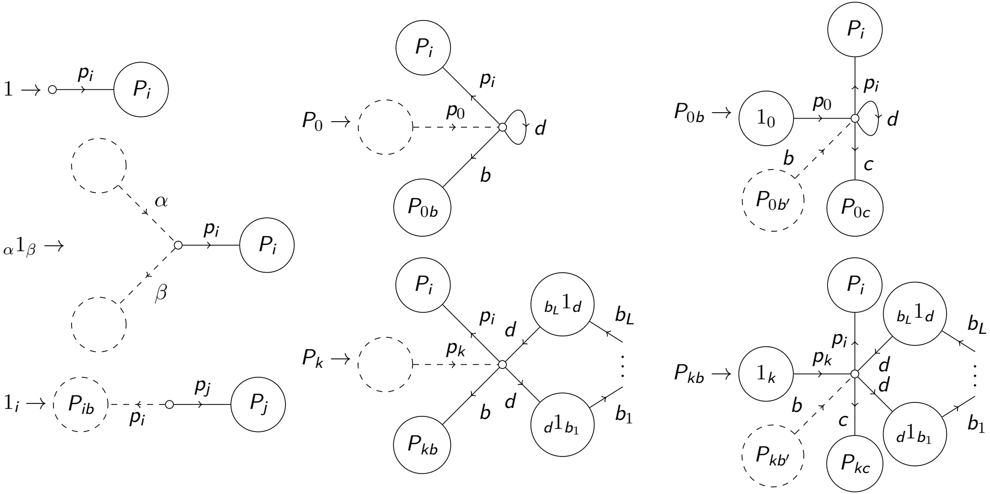
Generated Graph: vs "intermediate graphs"



A Grammar for $S\Gamma(1)$

Relations:





How do we Know that this Indeed Generates $S\Gamma(1)$?

Theorem (Stephen; 1990)

$M = \mathrm{Inv}\langle A \mid \lambda_i = 1 \rangle$ $\quad e \in A^{\pm *}$ *with* $e^2 = e$ *in* M $\quad \Gamma : A^{\pm 1}$*-labeled directed graph with root* q

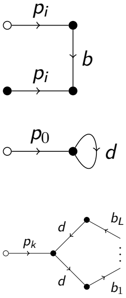
$\Gamma \simeq S\Gamma(e) \iff$

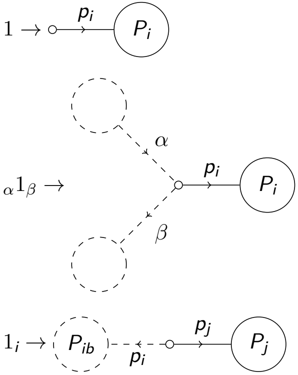
1. Γ *is symmetric, strongly connected, deterministic* ⇝ *check neighborhoods*
2. $e \in \mathscr{L}(q, q)$ ⇝ *trivial for* $e = 1$
3. $\forall p \in \Gamma : \lambda_i \in \mathscr{L}(p, p)$ ⇝ *check extended neighborhoods*
4. $\mathscr{L}(q, q) \subseteq \mathscr{U}(e) = \{u \in A^{\pm *} \mid u \geq e \text{ in } e\}$
   ⇝ This is the tricky part!

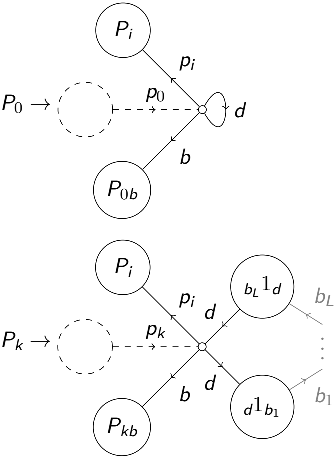
In our grammar: the neighborhood is fully determined by the nonterminal!
This neighborhood characterization also helps us to show that there are no automorphism.
Note: We only have to show that the root must be mapped to itself.

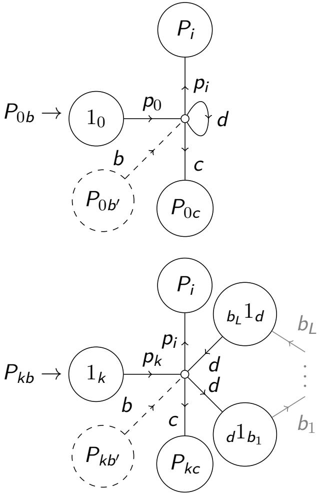
Again: The Grammar for $S\Gamma(1)$

Relations:









deterministic ✓

relations readable ✓

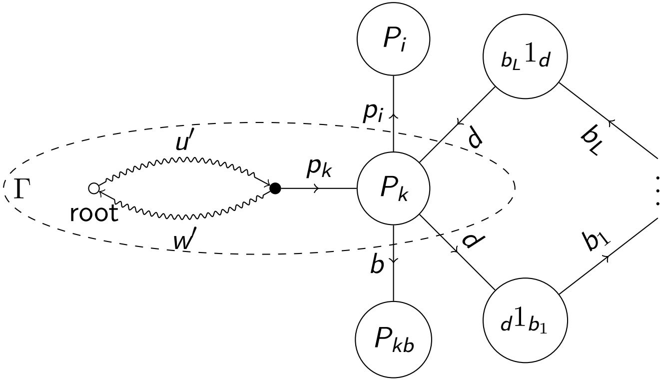
images of the root ✓

only 1 or $1_i \rightsquigarrow$ “$p_i b^{-1}$”

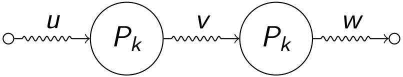
Upward Closure

- It remains to show: $\mathscr{L}(q, q) \subseteq \mathscr{U}(1) = \{u \in A^{\pm *} \mid u \geq 1 \text{ in } e\}$
- Formally, we define the generated graph Γ^* as the direct limit of the intermediate graphs Γ.
- Thus: It suffice to show the inclusion for all intermediate graphs!
- This allows for an inductive argument:
  Assume: Γ turns into Γ' in one step and $\mathscr{L}(\Gamma) \subseteq \mathscr{U}(1)$
  To show: $\mathscr{L}(\Gamma') \subseteq \mathscr{U}(1)$

Sketch of the Induction



- Let x label a circle at the root.
- If it lies completely in Γ, we have $x \in \mathscr{U}(1)$ by induction.
- Otherwise, factorize it at P_k:

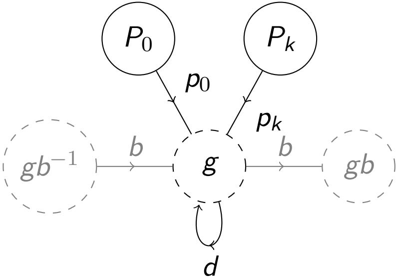


- W.l.o.g.: no other P_k visits
- We know: $u = u' p_k$ and $w = p_k^{-1} w'$
- Options for v
  - $v = p_i p_i^{-1}$
  - $v = bb^{-1}$
  - $v = db_1 \dots b_L d$
  - $v = db_1 \dots b_i b_i^{-1} \dots b_1^{-1} d^{-1}$

- $x = uvw = u' \underbrace{p_k d b_1 \dots b_l p_k^{-1}}_{=1} w' = u' w' \in \mathscr{U}(1)$

What About the Grammar for $S\Gamma(e)$?

- We re-use the grammar for $S\Gamma(1)$.
- This time we don't start with a single node but with the Cayley graph of G.
- Add the appropriate "decorations" to each node:



- Then: show the same things as for $S\Gamma(1)$...

Thank you!